

TESIS CARRERA DE DOCTORADO EN FÍSICA

**ASPECTS OF ENTANGLEMENT ENTROPY IN
ALGEBRAIC QUANTUM FIELD THEORY**

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*“329 pages,
not great, not terrible.”*

Comrade Anatoly Dyatlov

Resumen

En esta tesis, estudiamos aspectos de la entropía de entrelazamiento en teorías cuánticas de campos siguiendo un enfoque algebraico. La principal motivación del trabajo es explorar y comprender mejor la estructura general del entrelazamiento en teorías de campos relativistas, con el objetivo último de buscar una formulación axiomática de la teoría cuántica de campos en términos de cantidades puramente entrópicas y elementos de la teoría de información. Por otro lado, también estamos interesados en explorar las consecuencias del entrelazamiento en teoría algebraica de campos, con el propósito de descubrir propiedades no conocidas de las teorías cuánticas de campos y entender aún mejor las ya conocidas. Esto nos ayudaría con el propósito de encontrar un “principio dinámico” que nos permita construir, de forma rigurosa, modelos no triviales de teorías de campos relativistas. Como veremos a lo largo de esta tesis, el enfoque algebraico es el esquema natural para definir y estudiar el entrelazamiento en teoría cuántica de campos, y por lo tanto, para plantear y realizar estas investigaciones.

En los primeros capítulos de la tesis realizamos una revisión autocontenida de la teoría algebraica de campos y la teoría de información cuántica, mientras que en los capítulos finales nos centramos en los resultados obtenidos. Entre ellos, destacamos los cálculos, de forma matemáticamente rigurosa, de medidas de entrelazamiento y Hamiltonianos modulares para algunos modelos específicos de teorías de campos, usando técnicas de la teoría algebraica de campos y la teoría modular de álgebras de von Neumann. Los resultados obtenidos nos muestran explícitamente aspectos no locales y universales de los Hamiltonianos modulares, y nos ayuda a resolver ambigüedades que aparecen cuando uno realiza cálculos similares usando técnicas y métodos no rigurosos. También estudiamos, de forma general, las consecuencias e implicancias de la entropía de entrelazamiento en teorías de campos que presentan una estructura no trivial de sectores de superselección proveniente de simetrías globales. Para esto seguimos el enfoque algebraico desarrollado por Doplicher, Haag y Roberts. Como resultado, encontramos un parámetro de orden entrópico que “mide” el tamaño del grupo de simetría y el cual está formado por la diferencia de dos informaciones mutuas. Más aún, logramos identificar, para una teoría de campos general, cuáles son los operadores responsables de esta diferencia, y encontramos una relación desconocida (que involucra dichos operadores) de la teoría de información en teoría de campos: la relación

de certidumbre entrópica. También argumentamos que esta relación puede ser extendida a contextos más generales y mantiene una relación estrecha con la teoría de subfactores de álgebras de von Neumann.

Palabras clave: TEORÍA ALGEBRAICA DE CAMPOS, ÁLGEBRAS DE VON NEUMANN, ENTROPÍA DE ENTRELAZAMIENTO, HAMILTONIANOS MODULARES, SECTORES DE SUPERSELECCIÓN

Abstract

In this thesis, we study aspects of entanglement theory of quantum field theories from an algebraic point of view. The main motivation is to gain insights about the general structure of the entanglement in QFT, towards a definition of an entropic version of QFT. In the opposite direction, we are also interested in exploring any consequence of the entanglement in algebraic QFT. This may help us to reveal unknown features of QFT, with the final aim of finding a dynamical principle which allows us to construct non-trivial and rigorous models of QFT. The algebraic approach is the natural framework to define and study entanglement in QFT, and hence, to pose the above inquiries.

After a self-contained review of algebraic QFT and quantum information theory in operator algebras, we focus on our results. We compute, in a mathematically rigorous way, exact solutions of entanglement measures and modular Hamiltonians for specific QFT models, using algebraic tools from modular theory of von Neumann algebras. These calculations show explicitly non-local features of modular Hamiltonians and help us to solve ambiguities that arise in other non-rigorous computations. We also study aspects of entanglement entropy in theories having superselection sectors coming from global symmetries. We follow the algebraic perspective of Doplicher, Haag, and Roberts. In this way, we find an entropic order parameter that “measures” the size of the symmetry group, which is made out of a difference of two mutual informations. Moreover, we identify the main operators that take account of such a difference, and we obtain a new quantum information quantity, the entropic certainty relation, involving algebras containing such operators. This certainty relation keeps an intrinsic connection with subfactor theory of von Neumann algebras.

Keywords: ALGEBRAIC QFT, VON NEUMANN ALGEBRAS, ENTANGLEMENT ENTROPY, MODULAR HAMILTONIANS, SUPERSELECTION SECTORS

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List of symbols

a.e.: almost everywhere
BF: Buchholz-Fredenhagen
BW: Bisognano-Wichmann
CFT: conformal field theory
CFTs: conformal field theories
DHR: Doplicher-Haag-Roberts
EE: entanglement entropy
EEs: entanglement entropies
l.h.s.: left-hand side
MI: mutual information
MIs: mutual informations
QIT: quantum information theory
QFT: quantum field theory
QFTs: quantum field theories
qubit: quantum bit
RE: relative entropy
REs: relative entropies
r.h.s.: right-hand side
RS: Reeh-Schlieder
SS: superselection sector
vN: von Neumann

Chapter 1

Introduction

Quantum field theory (QFT) is the physical theory that emerged when physicists tried to construct a quantum theory that would be compatible with special relativity.¹ In other words, it is the combination of two physical concepts: quantum mechanics (QM) + special relativity. In the usual approach to QFT, both concepts are unified in the notion of a quantum field. On one hand, a quantum field is an operator acting on some physical Hilbert space (quantum physics), whereas on the other hand, it satisfies locality and transformation properties compatible with the symmetries of the spacetime (special relativity). Algebraic quantum field theory (AQFT) settles as an alternative way to unify these two concepts. More appropriately, the term “algebraic” in AQFT refers to the quantum physics part.

Algebraic quantum theory could be thought of as an extension of the usual approach to quantum mechanics [1, 2]. In the later, one first specified a Hilbert space \mathcal{H} and recognized the set of pure states as the unit rays in \mathcal{H} . Convex combinations of pure states represent mixed states, and they are in one-to-one correspondence with the statistical operators (density matrices) in the Hilbert space. Any self-adjoint operator in \mathcal{H} represents an (ideal) observable, and we are able, in principle, to construct any operator of the theory. However, in any real situation, we do not have at our disposal all the operators of theory. This may happen because of experimental constraints. For example, in our laboratory, we usually do not have the equipment to perform any measurement with arbitrary precision.

Whatever the circumstances may be, the mathematical way to describe this situation is limiting the operator content of the theory as a subset of all operators in the Hilbert space. Following physical motivation, we can argue that such a subset is indeed an algebra (a subalgebra of the algebra of all operators). More precisely, the quantum system is generally described by a C^* or von Neumann (vN) algebra \mathfrak{A} , where the algebra of all (bounded) operators in some Hilbert space is a particular example. Such an algebra is usually called the algebra of observables. The elements of \mathfrak{A} are the operators of the theory,

¹Along this thesis, the term QFT refers to a local Poincaré covariant quantum field theory. We exclude from our analysis non-relativistic or non-local quantum field theories.

and the self-adjoint ones represent observables. In this approach there is no prior reference to a Hilbert space, the algebra is considered as an abstract object, and the states are just linear functionals on the algebra. The number obtained by evaluating an observable on a state has to be interpreted as the expectation value and/or transition probability. Hilbert space arises when one considers a representation of the algebra. Contrary to what happens in the usual approach to QM, there exists many unitarily inequivalent representations of the observable algebra. One of the successes of the algebraic approach was to show that all the inequivalent representations represent, in the end, the same physical situation. Or in other words, every physical phenomenon describable in one representation could be also be described in any other representation. That means that the physical information is encoded in the algebra of observables itself. However, the choice of a particular representation has very computational importance. A physical process may have a simple description in a given representation whereas its description in other representations could be highly cumbersome. This is in fact what happens in QFT. The algebra of QFT admits many unitarily inequivalent representations. Among them, the Fock representation is very accurate to describe free fields but not for interacting fields. When one uses the Fock representation to make computations about interacting fields, we obtain divergent quantities everywhere. This is nothing more than the consequence of Haag's theorem that asserts that the interaction representation does not exist in QFT. This problem could be bypassed by introducing a cutoff and renormalizing the theory, to obtain analytical results that are in high accordance with physical experiments. However, from the conceptual and mathematical standpoint, this renormalization program is far from being satisfactory.

In the usual approach to QFT, the theory is described in a Hilbert space \mathcal{H} and the operator content is described by field operators $\phi(x)$. The local physics around a finite region \mathcal{O} of the spacetime is described by the field operators $\phi(x)$ with $x \in \mathcal{O}$. It is well-known that all such operators commute with the field operators $\phi(y)$ whenever y is spacelike separated with all points in \mathcal{O} . That means, that the operators "available" within the region \mathcal{O} are far from being the set of all operators in the Hilbert space \mathcal{H} . Then, they must be described by a proper subset $\mathfrak{A}(\mathcal{O})$ of operators, depending on the region \mathcal{O} . It is for this reason that the algebraic approach to quantum theory naturally fits for QFT. Furthermore, the algebraic approach to quantum theory typically arises interest when one wants to study QFT from an axiomatic point of view. The strong mathematical and conceptual basis of the theory of operators algebras allows us to establish a solid mathematical scenario for QFT, from which general statements could be proven. From this perspective, we can remark the algebraic proofs of the CPT and spin-statistics theorems, and the famous connection between superselection sectors and internal (global and local) symmetries. On the other hand, it has been very difficult to exhibit concrete models satisfying the axioms beyond the free theories, $d = 2$ CFTs, and some superrenormalizable theories.

The theory of von Neumann algebras could be thought of as a non-commutative generalization of the usual classical probability theory of Kolmogorov. More concretely, it can be shown that any classical probability space could be regarded as an Abelian (commutative) von Neumann algebra. The origin and development of telecommunications, classical computation, and data processing in the last century gave place to a strong interest in the study of statistical and information measures in (classical) probability spaces. It was Shannon who first interpreted the concept of entropy as information, giving place to the origin of classical information theory. Afterward, the concept of entropy in classical information theory is named Shannon entropy. Almost simultaneously, the theoretical proposal of the quantum computer by R. Feynman and P. Benioff gave birth quantum information theory (QIT). Like its classical counterpart, the mathematical formulation of QIT is described using “quantum” probability spaces, i.e. von Neumann algebras. It is for that reason that QIT theory matches perfectly with the algebraic approach to quantum physics. From the physical point of view, the central difference between classical and quantum physics is that quantum systems exhibit a distinct kind of statistical correlations (called generically entanglement) that are absent in classical systems. This significant difference has been used to show (at least theoretically) the potential ability of quantum computing devices to solve problems that classical computers practically cannot, leading to the notion of quantum supremacy in quantum computing. These correlations, due to the entanglement, can be captured with the help of the von Neumann entropy, the quantum version of the Shannon entropy, and for this reason, it is usually called entanglement entropy (EE).

However, the interest in studying statistical measures in quantum (or classical) systems does not only rely on information or computation theories. From a physical standpoint, such an interest relies on the study of bulk or macroscopic quantities of systems governing a large number of individual degrees of freedom. This explains the fact that QIT (and the theory of vN algebras) has much more to do with QFT (or many-body systems) than with the usual Schrodinger’s mechanics theory. In condensed matter physics, the entanglement entropy has been used as an order parameter to distinguish phases that could not be identified by symmetry [3, 4], and also to localize and characterize critical points in phase transitions [5–8]. In quantum gravity, more specifically in the context of holography, there is a deep connection between entanglement entropy and geometry, materialized in the Ryu-Takayanagi formula [9]. This formula generalizes the usual expression for the black hole entropy, and it has helped us to better specify the holographic dictionary in AdS/CFT [10–13]. In particular, the continuity and emergence of spacetime could be explained in terms of the entanglement of the boundary degrees of freedom [14]. In QFT, the EE of the vacuum state holds the distinct feature that it grows as the area of the region [15–17]. The first insight of this area law was observed while trying to explain the statistical origin of the entropy of black holes as an EE [18–20]. From the algebraic perspective, given a vN

algebra representing some physical system and a state on such an algebra, there always exists a natural dynamics on the system, depending on the state, even if the original system has no preferred dynamics [2, 21]. This evolution is called modular flow, and its Hermitian generator is called modular Hamiltonian. Of course, the conceptual similarity of this feature with gravitational physics is almost evident, and it has been exploited in the AdS/CFT scenario to improve the holographic dictionary [22, 23]. Moreover, the modular Hamiltonian is also used in the definition of the relative entropy on general vN algebras [24, 25].

Needless to say, the interest of EE in high energy physics is beyond quantum gravity. In fact, in QFT, it has an interest in its own right. The EE has been exploited to give “entropic” proofs of the renormalization group irreversibility theorems in $d = 2, 3$, and 4 [26–32]. There also exists a deep connection between spacetime symmetries and modular evolutions for some particular types of regions. This is the famous theorem of Bisognano-Wichmann, which asserts that, for any QFT, the modular dynamics for wedge regions and the vacuum state coincides with the one-parameter of boost transformations that leave the wedge region invariant [33]. For CFTs, this could be generalized for double cones. In that case, the modular evolution is given by the one-parameter group of conformal transformations that leave the double cone invariant [34, 35]. However, for other types of regions, this feature is lost, and even more, the modular evolution is no longer local [36, 37]. There have been important efforts to understand these non-local features [37, 38], and in particular, to compute modular Hamiltonians and entanglement measures for concrete models and more general regions [36, 39, 40].

There is another formal aspect that relates QIT with QFT. The Wightman axiomatic approach of QFT has an equivalent description in terms of vacuum correlators [41]. As it is well-known, the Wightman axiomatic approach of QFT has an equivalent description in terms of vacuum correlators. The knowledge of all such correlators allows us to reconstruct the theory uniquely. In other words, a complete set of field vacuum expectation values, satisfying a certain set of axioms, gives place to a unique QFT (up to unitary equivalence) satisfying the Garding-Wightman axioms. In AQFT, the physical content of the theory is no-longer codified in a point-like localized field $\phi(x)$, but in a collection of algebras $\mathfrak{A}(\mathcal{O})$, assigned to spacetime regions \mathcal{O} , satisfying a set of axioms that take into account the algebraic and local properties of the fields. The analogous description of quantum fields in terms of correlators would be a map that assigns to each local algebra (or equivalently to each spacetime region) a number that “quantifies” the collective behavior of the degrees of freedom inside the region \mathcal{O} . A natural candidate is the EE of such regions. However, the EE is UV divergent in QFT, and hence, it needs a cutoff to be defined, making this quantity ambiguous in the continuum QFT. This problem could be solved invoking the mutual information, which is finite and well-defined quantity for any pair of disjoint regions. We

can imagine that the knowledge of all mutual information $I(\mathcal{O}_1, \mathcal{O}_2)$ for all pairs of disjoint regions \mathcal{O}_1 and \mathcal{O}_2 in the vacuum state should be enough to reconstruct the underlying QFT model uniquely. This would result in an entropic formulation of QFT, strongly supporting the idea that “information” is a very fundamental concept in (quantum) physics. Towards this aim, we need to reach deeper knowledge about the behavior of EE and entanglement measures in QFT.

The aim of this thesis is to make some contributions along the lines discussed in the last two paragraphs. In particular, we study exact and mathematical rigorous computations of entanglement measures and modular Hamiltonians in specific models of QFT from the algebraic perspective. We also study formal aspects of the entanglement theory in AQFT. The main results of this work are the following.

1. To begin with, we compute the relative modular Hamiltonian and the relative entropy for coherent states and the Rindler wedge for the theory of a free scalar [42]. Previous non-rigorous calculations, including path integral methods and computations from the lattice, give a result for such a relative entropy, which involves integrals of expectation values of the energy-momentum stress tensor along the considered region. However, the non-uniqueness of the stress tensor leads to an ambiguity in the modular Hamiltonian, which manifests in the usual formula for the relative entropy of coherent states. Then, the main motivation for the computation of such a relative entropy is to solve this puzzle.
2. In the same lines, we compute the vacuum modular Hamiltonian and the vacuum mutual information for a two-interval region and the theory of a free chiral current. As in the case of a free chiral fermion, the modular Hamiltonian turns out to be non-local, but in this case, the non-local term is given by a smooth kernel. Contrary to the free chiral fermion, this model shows a failure of duality for two intervals that translates into a loss of a symmetry property for the mutual information usually associated with modular invariance. Moreover, we find explicitly the operators responsible for the failure of the duality condition. These operators commute with all the operators of the two intervals, but they do not belong to the algebra of the complement region of the two intervals. Such operators naturally appear when the theory has non-trivial superselection sectors (SS).
3. The above observation motivated us to study the consequences of the superselection structure in the EE from a general perspective. Following the algebraic approach to sector theory, we find a novel connection between EE and internal global symmetries. In this way, we find that the mutual information between spacelike regions encodes information about the superselection structure of the theory. More precisely, we find an entropic order parameter, constructed by a difference of two mutual in-

formations, that measures the size of the global symmetry group, which accounts for the superselection structure of the theory. Moreover, we identify the main operators that take account of such a difference, and we find an entropic certainty relation involving relative entropies on algebras containing such operators. We argue that this entropic certainty relation may have a deep connection with the theory of vN subfactors, and it may quantify the algebraic μ -index of the inclusion of algebras [43, 44]. We complement this analysis by applying the mentioned entropic order parameter to different physical situations, such as finite and Lie symmetry groups, compactified scalars, regions with different topologies, excited states, and spontaneous symmetry breaking.

We expect that all these contributions will help to better understand the possible correspondence between AQFT and QIT.

The content of the thesis is organized as follows. In chapter 2, we review the algebraic approach to quantum field theory. To separate the algebraic features from the ones related to locality and covariance, we divided the chapter into two sections. The first one is dedicated to algebraic quantum theory alone, and in the second one, we explain how the content of the first section is used to describe QFT from an algebraic perspective. In chapter 3, we review the theory of quantum information in general operator algebras, with particular emphasizes on entanglement and information measures. For a better discussion, we treat first the case of finite quantum systems (finite-dimensional algebras) and then the case of general quantum systems (infinite-dimensional algebras). In chapter 4, we combine the ideas developed in the two previous chapters to review the basic aspects of EE in QFT. We put special emphasizes on how to describe entanglement in the algebraic framework, using tools that are well-defined in the continuum QFT. The reader who is familiar with these concepts may skip these chapters and go directly to chapter 5, where we study the relative entropy for coherent states for the theory of a free scalar field. The computation of such a relative entropy is performed in a complete algebraic and mathematically rigorous way, and it helps us to solve the discussed ambiguities on the modular Hamiltonian, which arise in computations using non-rigorous methods. In chapter 6, we compute explicitly the vacuum modular Hamiltonian and the mutual information for two intervals for the theory of a chiral free scalar field. We determine explicitly the eigenvectors that diagonalized the correlator kernel using a novel method of finding complex analytic functions obeying specific boundary conditions. We first also apply this method to rederive the vacuum modular Hamiltonian for a free chiral fermion in multiple intervals, previously studied in [36]. In chapter 7, we study aspects of entanglement entropy in theories having superselection sectors. We focus on theories with sectors arising from global symmetries following the algebraic approach of Doplicher, Haag, and Roberts [45–50]. We end in chapter 8 with a summary and the conclusions of the work.

Chapter 2

Algebraic approach to QFT

The main purpose of this chapter is to define the concept of an algebraic quantum field theory (AQFT). As we explain in the introduction, QFT is the combination of two physical concepts: quantum mechanics and special relativity. The term “algebraic” in AQFT refers to the first of such concepts. In fact, the first section of the chapter is devoted to explaining what an algebraic quantum theory is, and in which sense it generalizes the usual approach to quantum mechanics. We first define the concept of C^* -algebra, usually called algebra of observables, which is the mathematical object used to describe quantum systems. After we introduce the notion of states and representations, we proceed to study general consequences of the theory. Among them, we remark the concept of physical equivalence between representations and a theorem that asserts that all representations are physically equivalent. In other words, it means that the physical content of the theory is encoded in the algebra of observables itself, rather than in the representation used for calculations. We also study the structure of the space of (pure) states of the theory, which naturally decomposes into coherent subsets or sectors. Consequently, we show a one-to-one correspondence between irreducible representations of the algebra of observables and sectors of pure states. At the end of the first section, we introduce the notion of von Neumann algebra, which is a mathematical concept closely related to C^* -algebra.

In the second section of this chapter, we turn to study AQFT itself. First of all we state the axioms of AQFT, followed by the concept of vacuum states and vacuum representations. This last concepts allow us to state the famous theorem of Reeh-Schlieder. After all the formal aspects are explained, we give some comments about the relation between this approach to QFT and the usual axiomatic approach of Garding and Wightman. Then, we explain how AQFT incorporates the notion of fermionic operators, which are excluded in our first definition of QFT. The last part of this second section is dedicated to the study of the lattice structure of AQFT, where the concept of duality for local algebras plays a salient role.

2.1. Algebraic quantum theory

In the usual approach to quantum theory, the mathematical description of the system uses a Hilbert space \mathcal{H} and the set of its bounded linear operators $\mathcal{B}(\mathcal{H})$.¹ The set of all observables coincides with the subset $\mathcal{B}_{sa}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ of all self-adjoint operators, and the set of states coincides with the space of density matrices, i.e.

$$\mathfrak{S}(\mathcal{H}) := \{\rho \in \mathcal{B}(\mathcal{H}) : \rho \geq 0 \text{ and } \text{Tr}(\rho) = 1\} . \quad (2.1)$$

In fact, any density matrix ρ gives place to a linear functional $\omega_\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ through the relation

$$\omega_\rho(A) := \text{Tr}(\rho A) , \quad \text{for all } A \in \mathcal{B}(\mathcal{H}) . \quad (2.2)$$

The physical interpretation of (2.2) has two equivalent meanings. Suppose that we are able to perform a physical experiment where we measure the physical observable represented by $A \in \mathcal{B}_{sa}(\mathcal{H})$ in the physical state represented by $\rho \in \mathfrak{S}(\mathcal{H})$.

1. Equation (2.2) corresponds to the expectation value of the physical observable represented by A in the physical state represented by ρ . such an expectation value coincides with the average of the results of the measurement experiment after the experiment is repeated N times, in the limit when N is large enough.
2. Due to the spectral theorem, any self-adjoint operator A can be uniquely decomposed as

$$A = \int_{\mathbb{R}} \lambda dE_A(\lambda) , \quad (2.3)$$

where $E_A(\Delta)$ are their spectral measures corresponding to the measurable set $\Delta \subset \mathbb{R}$. Then, the expectation value

$$p_A(\Delta) := \omega_\rho(E_A(\Delta)) = \text{Tr}(\rho E_A(\Delta)) , \quad (2.4)$$

represents the probability that the measured value lies in the set Δ .

Remark 2.1. We do not lose any generality considering only bounded observables. Indeed, any unbounded self-adjoint X operator can be decomposed as in (2.3), and its spectral measures $E_X(\Delta) \in \mathcal{B}(\mathcal{H})$ whenever $\Delta \subset \mathbb{R}$ bounded. In the practice, when we measure any observable associated with an unbounded operator X (e.g. the energy, which is associated

¹A bounded operator $A \in \mathcal{B}(\mathcal{H})$ is a linear map $A : \mathcal{H} \rightarrow \mathcal{H}$ for which exists a constant $K_A \geq 0$ such that $\|A|\Psi\rangle\| \leq K_A \|\Psi\|$ for all $|\Psi\rangle \in \mathcal{H}$. This last formula allows to define the norm of such an operator as $\|A\| := \sup\{\|A|\Psi\rangle\| : \|\Psi\| = 1\}$. The outstanding property about bounded operators is that they are well-defined over the whole Hilbert space, whereas an unbounded operator X has as a domain a proper subset of $\mathcal{D}(X) \subsetneq \mathcal{H}$, depending on X . This implies that the composition (product) of two bounded operators is always well-defined, whereas the composition of unbounded operators is not in general well-defined because of problems concerning domains. For further details see [51].

with the Hamiltonian), we employ a measurement device, which has some “finite measurement range” (e.g. the interval between minimum and maximum value of energy that the instrument is capable to detect). Measurement devices with infinite measurement range do not exist.² According to (2.3), this means that, in practice we measure the observable associated with the bounded operator

$$X = \int_{x_{min}}^{x_{MAX}} \lambda dE_A(\lambda) . \quad (2.5)$$

Remark 2.2. The correspondence principle, i.e. the identification of physical observables and physical states with the mathematical objects in $\mathcal{B}_{sa}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H})$, is a very difficult theoretical-experimental task that has to be solved for any particular physical situation. A general solution to this “correspondence problem” has not been achieved, and it is one of the open problems in the foundations of quantum mechanics. This problem is beyond the purpose of this thesis. Here, we always start with a mathematical set up as above (or with any of its generalizations as we argue below), which may describe some physical system.

Consider the physical situation when we do not have all the observable operators, i.e. all the set $\mathcal{B}_{sa}(\mathcal{H})$, to our disposal. How does the mathematical description of the physical system change? The algebraic approach to quantum theory borrows with the purpose to give a formal description to this situation. In fact, the solution is quite simple: the observables are described by the self-adjoint elements of some subset $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$, and the states are just a specific subset of the space of linear functionals over \mathfrak{A} . From the mathematical point of view, such a subset \mathfrak{A} is described by a C^* -algebra, which is the starting point for the next subsection.

Remark 2.3. The situation explained above covers what usually happens in QFT. There, the local physics around a finite region \mathcal{O} of the spacetime, is described by field operators $\phi(x)$ with $x \in \mathcal{O}$. It is well-known that all such operators commute with the (non-trivial) field operators $\phi(y)$ whenever y is spacelike separated with all points in \mathcal{O} . That means that the operators “available” within the region \mathcal{O} are far from being the set of all operators in the Hilbert space \mathcal{H} . They must be described by a proper subset $\mathfrak{A}(\mathcal{O}) \subsetneq \mathcal{B}(\mathcal{H})$. It is for this reason that the algebraic approach to quantum theory naturally fits for QFT.

2.1.1. C^* -algebras

It is well-known that the set of all bounded operators $\mathcal{B}(\mathcal{H})$ has a natural structure of algebra: it is closed under linear combinations and multiplication (composition of operators). However, the subset $\mathcal{B}_{sa}(\mathcal{H})$ is not an algebra because the product of two self-adjoint operators is not, in general, a self-adjoint operator. Furthermore, the algebra generated by

²In the theory, we used them as idealized observables.

$\mathcal{B}_{sa}(\mathcal{H})$ is indeed $\mathcal{B}(\mathcal{H})$, because any operator $A \in \mathcal{B}(\mathcal{H})$ is a linear combination of two self-adjoint operators: $A = \frac{A+A^*}{2} + i\frac{A-A^*}{2i}$. For all that, in the usual approach to quantum theory, it is better to consider $\mathcal{B}(\mathcal{H})$ as the primary mathematical object describing the operator content of the theory, and then, the observables are defined as the self-adjoint elements of $\mathcal{B}(\mathcal{H})$.

As we explained above, in the case where not all operators are available, the theory is described by a subset of $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$. From the operational point of view we expect that, if two operators $A, B \in \mathcal{B}(\mathcal{H})$ are available for measurements/operations on the system, i.e. $A, B \in \mathfrak{A}$, then, any linear combination ($\lambda_1 A + \lambda_2 B$ with $\lambda_1, \lambda_2 \in \mathbb{C}$) and the product (AB) of such operators, also belong to \mathfrak{A} . These imply that the subset \mathfrak{A} should be a subalgebra of $\mathcal{B}(\mathcal{H})$. This is one of our first conclusions: a general quantum system is described by a subalgebra \mathfrak{A} of $\mathcal{B}(\mathcal{H})$, or in a more abstract sense, a general quantum system is described by an algebra \mathfrak{A} , which is not in general isomorphic³ to the algebra $\mathcal{B}(\mathcal{H})$ of all operators in a Hilbert space.

The set $\mathcal{B}(\mathcal{H})$ has further algebraic and topological properties, which makes it a C^* -algebra. These properties are expected to be satisfied for any of its subalgebras, which describe physical systems. Such properties motivate the following definitions.

Definition 2.4. Let \mathfrak{A} be a complex (abstract⁴) algebra.⁵ An antilinear map $*$: $\mathfrak{A} \rightarrow \mathfrak{A} / A \mapsto A^*$ is called an *involution* if $A = (A^*)^*$ and $(AB)^* = B^*A^*$ for all $A, B \in \mathfrak{A}$. An algebra with an involution is called a **-algebra* or *involution algebra*.

Definition 2.5. Let \mathfrak{A} be a complex (abstract) algebra. A map $\|\cdot\| : \mathfrak{A} \rightarrow \mathbb{R}_{\geq 0} / A \mapsto \|A\|$ is called a *norm* if it satisfies

1. $\|A\| = 0$ if and only if $A = 0$,
2. $\|\lambda A\| = |\lambda| \|A\|$,
3. $\|A + B\| \leq \|A\| + \|B\|$,
4. $\|AB\| \leq \|A\| \|B\|$,

for all $A, B \in \mathfrak{A}$ and all $\lambda \in \mathbb{C}$. An algebra with a norm is called a *normed algebra*.

Definition 2.6. Let \mathfrak{A} be a normed algebra. The topology arising from the norm is called the *uniform* or *norm topology*. A normed algebra complete in the uniform topology is called a *Banach algebra*.

³See definition 2.12.

⁴The term abstract is placed there to emphasize that the algebra may not be a concrete subalgebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ of operators acting on some Hilbert space \mathcal{H} .

⁵All the algebras considered along this thesis are *unital* algebras, which means that exists $\mathbf{1} \in \mathfrak{A}$ such that $\mathbf{1} \cdot A = A \cdot \mathbf{1} = A$ for all $A \in \mathfrak{A}$.

Definition 2.7. Let \mathfrak{A} be an involution Banach algebra. \mathfrak{A} is said to be a C^* -algebra if the norm satisfies $\|A^*A\| = \|A\|^2$ for all $A \in \mathfrak{A}$ (C^* -property).

All the properties described in the previous definition are satisfied by $\mathcal{B}(\mathcal{H})$, which give place to the following lemma.

Lemma 2.8. For any Hilbert space \mathcal{H} , the set of all bounded linear operators $\mathcal{B}(\mathcal{H})$ is a C^* -algebra. The norm is given by the operator norm⁶ and the involution is given by taking adjoints.

Proof. See [21]. □

The above lemma gives an abstract characterization of the set of operators in the usual approach to quantum theory. The algebraic approach to quantum theory generalizes the usual approach just claiming that the mathematical description of a quantum system is given by some C^* -algebra \mathfrak{A} , where the case $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ is just a particular case. In this context, \mathfrak{A} is called the *algebra of observables* and any element $A \in \mathfrak{A}$ is called an *operator*.

This generalization allows us to include in the theory the description of more general and sophisticated systems. As we argue above, this more general algebras are needed to describe the local physics in QFT in a rigorous mathematical way. Of course, the operators are such one part of the description of the theory, the other part corresponds to the states. We postponed the explanation of how the states are described in this algebraic setting for the following subsection.

Now, we borrow some definitions from the usual theory of operator algebras.

Definition 2.9. Let \mathfrak{A} be a C^* -algebra. An element $A \in \mathfrak{A}$ is said to be *self-adjoint* if $A = A^*$. An element $U \in \mathfrak{A}$ is said to be *unitary* if $U^* = U^{-1}$. An element $B \in \mathfrak{A}$ is said to be *positive* if exists $A \in \mathfrak{A}$ such that $B = A^*A$. An element $P \in \mathfrak{A}$ is said to be (orthogonal⁷) *projector* if $P = P^2 = P^*$.

With all these definitions, we recognize that the observables of the theory are described by self-adjoints operators in \mathfrak{A} . On the other hand, subalgebras of C^* -algebras are characterized by the following lemma.

Definition 2.10. Let \mathfrak{A} be a C^* -algebra, and let $\mathfrak{B} \subset \mathfrak{A}$ a $*$ -subalgebra in the pure algebraic sense (closed by linear combinations, products and involution) and closed in the uniform topology. Then, we say that \mathfrak{B} is a C^* -subalgebra of \mathfrak{A} , and of course, it is a C^* -algebra on its own.

⁶See footnote 1.

⁷Throughout this thesis, all projectors are orthogonal.

Among the class of all C^* -algebras, we have the subclass of *concrete* C^* -algebras, i.e. the subclass of all C^* -subalgebras of the form $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} . The term concrete refers to the fact that the elements of \mathfrak{A} are indeed operators acting on some Hilbert space \mathcal{H} . At first glance, it seems that the class of concrete C^* -algebra is just a proper subclass of the class of all C^* -algebras. However, there is a theorem that asserts that any abstract C^* -algebra can be viewed as C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} . This is summarized in the following.

Lemma 2.11. *Let \mathfrak{A}_1 and \mathfrak{A}_2 be C^* -algebras. Any morphism⁸ $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ is continuous (in the uniform topology), i.e. it is an homomorphism.*

Proof. See [21]. □

Definition 2.12. Let \mathfrak{A}_1 and \mathfrak{A}_2 be C^* -algebras. A bijective homomorphism $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ is called a C^* -isomorphism. Moreover, given any $\mathfrak{A}_1, \mathfrak{A}_2$ C^* -algebras, \mathfrak{A}_1 and \mathfrak{A}_2 are said to be C^* -isomorphic (and denoted by $\mathfrak{A}_1 \simeq \mathfrak{A}_2$) if there exists a C^* -isomorphism $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$.

Theorem 2.13. *Let \mathfrak{A} be a C^* -algebra. Then, there exists⁹ a Hilbert space \mathcal{H} and a C^* -subalgebra $\mathfrak{B} \subset \mathcal{B}(\mathcal{H})$ such that $\mathfrak{A} \simeq \mathfrak{B}$.*

Proof. See [21]. □

The above lemma shows that we do not lose any generality assuming that our quantum system is described by a concrete C^* -algebra. However, the description of the quantum system without reference to any Hilbert space, allows us to study general properties of the algebra of observables that are intrinsic to the algebra itself. The isomorphism of lemma 2.13 is just a representation of the abstract algebra in some Hilbert space. There are many representations of it, which are C^* -isomorphic but not unitarily equivalent. In the end, all representations describe the same physics. But the choice of one particular representation is a matter of convenience related to the physical process one is interested to study. A given physical process may admit a simple description in some representation, but, at the same time, the description of the same process may be highly complex in a different representation. We discuss all these issues in the following subsection.

In the algebraic approach, the dynamics of the system is implemented as follows.

Definition 2.14. Let \mathfrak{A} be a C^* -algebra of observables. A *dynamics* for \mathfrak{A} is a one-parameter family of automorphisms $t \in \mathbb{R} \mapsto \alpha_t \in \text{Aut}(\mathfrak{A})$. The pair (\mathfrak{A}, α_t) is called a C^* -dynamical system.

⁸A morphism $\Phi : \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ between $*$ -algebras is any linear map satisfying $\Phi(AB) = \Phi(A)\Phi(B)$ and $\Phi(A^*) = \Phi(A)^*$.

⁹Possibly non-separable.

Automorphisms are the algebraic version of unitary evolution. A particular kind of C^* -dynamical systems are the *closed systems*, which is described by a one-parameter group of automorphisms, i.e. $\alpha_{t_1} \circ \alpha_{t_2} = \alpha_{t_1+t_2}$ for all $t_1, t_2 \in \mathbb{R}$.

Example 2.15. (CCR and Weyl algebra) In Schrodinger's mechanics, the operator content of the theory is encoded in two basic operators: position and momentum. The *Canonical Commutation Relations algebra* (CCR algebra) for N degrees of freedom, is defined as the unital $*$ -algebra generated by the objects $q_1, \dots, q_n, p_1, \dots, p_n$ modulo the relations

$$[q_j, p_k] = i\delta_{jk}\mathbf{1}, \quad [q_j, q_k] = [p_j, p_k] = 0, \quad (2.6)$$

$$q_k = q_k^* \text{ and } p_k = p_k^*. \quad (2.7)$$

Here, we want to emphasize that we are considering the CCR algebra as an abstract algebra, independently of whether it admits a representation as a subalgebra of operators acting on some Hilbert space. We postpone such a problem for section 2.1.3. Unfortunately, it can be shown, that the CCR is not a C^* -algebra. In other words, it is impossible to equip the CCR algebra with a norm in order that all its elements have a finite norm and they satisfy all the algebraic relations.¹⁰ The canonical way to bypass this issue is to transform such self-adjoint elements into unitaries. For that, we defined the n -parameter groups of unitaries

$$W_q(\bar{a}) := e^{i\bar{q}\cdot\bar{a}} \text{ and } W_p(\bar{b}) := e^{i\bar{p}\cdot\bar{b}}, \quad (2.8)$$

where $\bar{a}, \bar{b} \in \mathbb{R}^n$. Formally, relations (2.6-2.7) are converted into

$$W_p(\bar{b}) W_q(\bar{a}) = e^{i\bar{a}\cdot\bar{b}} W_q(\bar{a}) W_p(\bar{b}), \quad (2.9)$$

$$W_q(\bar{a}_1) W_q(\bar{a}_2) = W_q(\bar{a}_1 + \bar{a}_2), \quad W_q(\bar{a}_1) W_q(\bar{a}_2) = W_q(\bar{a}_1 + \bar{a}_2), \quad (2.10)$$

$$W_q(\bar{a})^* = W_q(-\bar{a}) \text{ and } W_p(\bar{b})^* = W_p(-\bar{b}). \quad (2.11)$$

Any N -parameter family of unitaries $W_q(\bar{a}), W_p(\bar{b})$ satisfying (2.9-2.11) defines a *Weyl algebra* \mathcal{W}_n . The Weyl algebra encodes all the same information contained in the CCR algebra, but in a way that involves unitary (bounded) operators. It can be shown that the Weyl algebra \mathcal{W}_N can be equipped with a (unique) norm $\|\cdot\|$ in order to convert it into a C^* -algebra [52]. The Weyl algebra can be easily generalized to infinitely many degrees of freedom. Given a real Hilbert space \mathfrak{H} , the Weyl algebra $\mathcal{W}_{\mathfrak{H}}$, associated with \mathfrak{H} , is defined as the algebra generated by the unitary elements $W_\varphi(f), W_\pi(g)$ with $f, g \in \mathfrak{H}$, modulo

¹⁰In the context of representations, there is no representation of CCR algebra as bounded operators in a Hilbert space. If one tries to represent it, one finds that at least one of the elements q_k or p_k (for all k), must be represented by an unbounded operator.

the relations

$$W_\pi(g) W_\varphi(f) = e^{i\langle f, g \rangle_{\mathfrak{H}}} W_\varphi(f) W_\pi(g) , \quad (2.12)$$

$$W_\varphi(f_1) W_\varphi(f_2) = W_\varphi(f_1 + f_2) , \quad W_\pi(g_1) W_\pi(g_2) = W_\pi(g_1 + g_2) , \quad (2.13)$$

$$W_\varphi(f)^* = W_\varphi(-f) , \quad W_\pi(g)^* = W_\pi(-g) . \quad (2.14)$$

As in the finite case, the Weyl algebra $\mathcal{W}_{\mathfrak{H}}$ can be equipped with a (unique) C^* -norm [52]. When $\mathfrak{H} = L^2(\mathbb{R}^d, \mathbb{R})$, (2.8) suggests that we may heuristically write

$$W_\varphi(f) = e^{i \int \varphi(x) f(x) d^d x} \quad \text{and} \quad W_\pi(g) = e^{i \int \pi(x) g(x) d^d x} . \quad (2.15)$$

Performing formal manipulations to relations (2.12-2.14), we may arrive to

$$[\varphi(x), \pi(y)] = i\delta^{(d)}(x - y) \mathbf{1} , \quad [\varphi(x), \varphi(y)] = [\pi(x), \pi(y)] = 0 , \quad (2.16)$$

$$\varphi(x) = \varphi(x)^* \quad \text{and} \quad \pi(x) = \pi(x)^* . \quad (2.17)$$

2.1.2. States

Given a C^* -algebra \mathfrak{A} , expression (2.2) suggests that a state has to be described by a linear functional on the algebra \mathfrak{A} . We denote the dual space of \mathfrak{A} by \mathfrak{A}^* , i.e. \mathfrak{A}^* is the linear space of continuous¹¹ linear functionals over \mathfrak{A} . We define the norm of any functional $\phi \in \mathfrak{A}^*$ as

$$\|\phi\| := \sup \{ |\phi(A)| : \|A\| = 1 \} . \quad (2.18)$$

In fact, the space \mathfrak{A}^* equipped with such a norm is a Banach space. The functionals of particular interest are defined as follows.

Definition 2.16. Let \mathfrak{A} be a C^* -algebra. A linear functional $\omega \in \mathfrak{A}^*$ is defined to be *positive* if $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$, and it is defined to be *normalized* if $\|\omega\| = 1$. Any positive and normalized linear functional is called a *state*, and the set of all states is denoted by $\mathfrak{S}(\mathfrak{A})$. Any strictly positive state, i.e. $\omega(A^*A) > 0$ for all $A \in \mathfrak{A}$, is called a *faithful* state.

Remark 2.17. In the above definition, the condition of a state to be a continuous functional could be ignored. This is because, any positive linear functionals over C^* -algebra is continuous.

Remark 2.18. Any positive functional $\phi \in \mathfrak{A}^*$ is automatically *hermitian*, i.e. $\phi(A) \in \mathbb{R}$ for all $A \in \mathfrak{A}$ self-adjoint.

Remark 2.19. Positive functionals $\phi \in \mathfrak{A}^*$ also satisfy $\|\phi\| = \phi(\mathbf{1})$. Therefore, the normalization property of a state $\omega \in \mathfrak{S}(\mathfrak{A})$ is equivalent to $\omega(\mathbf{1}) = 1$.

¹¹For technical reasons, we only consider functionals that are uniform continuous.

The interpretation of states is the same as we explained in section 2.1. Given a state $\omega \in \mathfrak{S}(\mathfrak{A})$ and a self-adjoint operator $A \in \mathfrak{A}$, the real number $\omega(A)$ represents the expectation value of the physical observable represented by A in the physical state represented by ω . Similarly, given any projector P representing some logic proposition (e.g. “the result of measuring the physical observable represented by the self-adjoint operator A in the physical state represented by ω lies in the measurable set $\Delta \subset \mathbb{R}$ ”), $\omega(P)$ corresponds to the probability that such a proposition is true, and henceforth, $\omega(\mathbf{1} - P)$ corresponds to the probability that such a proposition is false.

If α is an automorphism of the C^* -algebra \mathfrak{A} and $\omega \in \mathfrak{S}(\mathfrak{A})$ a state, then $\omega \circ \alpha$ defined by $(\omega \circ \alpha)(A) := \omega(\alpha(A))$ is also a state. When $\mathfrak{B} \subset \mathfrak{A}$ a C^* -subalgebra, we can define the state $\omega|_{\mathfrak{B}} \in \mathfrak{S}(\mathfrak{B})$, called *the restriction of ω to \mathfrak{B}* , just by restricting the expectation values to the subalgebra, i.e. $\omega|_{\mathfrak{B}}(B) := \omega(B)$ for all $B \in \mathfrak{B}$. When there is no ambiguity, we denote $\omega|_{\mathfrak{B}}$ simply as ω .

In the usual approach to quantum theory, it is well-known that any convex combination of density matrices is a density matrix, and pure states are given just by unit rays in the Hilbert space. The algebraic counterpart of such facts is summarized in the following lemma and definition.

Lemma 2.20. *Let \mathfrak{A} be a C^* -algebra. The set of states $\mathfrak{S}(\mathfrak{A})$ is a convex set, i.e. $\lambda\omega_1 + (1 - \lambda)\omega_2 \in \mathfrak{S}(\mathfrak{A})$ whenever $\omega_1, \omega_2 \in \mathfrak{S}(\mathfrak{A})$ and $0 \leq \lambda \leq 1$.*

Proof. See [21]. □

Definition 2.21. Any extremal element of $\mathfrak{S}(\mathfrak{A})$ (i.e. any state which cannot be written as a non-trivial convex combination of any other two states) is a *pure state*. Any non-pure state is called a *mixed state*. The set of pure states is denoted by $\mathfrak{S}_p(\mathfrak{A})$.

Remark 2.22. The notions of pure and mixed states have no relation to the notion of subalgebras. Indeed, if $\mathfrak{B} \subset \mathfrak{A}$ is a C^* -subalgebra of \mathfrak{A} and $\omega \in \mathfrak{S}(\mathfrak{A})$ is a state, then $\omega|_{\mathfrak{B}} \in \mathfrak{S}(\mathfrak{B})$ could be pure or mixed independently of whether $\omega \in \mathfrak{S}(\mathfrak{A})$ is pure or mixed.

Example 2.23. For $\mathcal{B}(\mathcal{H})$, the set of pure states is given by

$$\mathfrak{S}_p(\mathcal{B}(\mathcal{H})) := \{\omega(\cdot) := \text{Tr}(|\Psi\rangle\langle\Psi|\cdot) \text{ with } |\Psi\rangle \in \mathcal{H} \text{ and } \|\Psi\| = 1\}, \quad (2.19)$$

i.e. it is in one-to-one correspondence with the set of unit rays in the Hilbert space \mathcal{H} .

Definition 2.21 gives a precise definition of what a pure state is in the algebraic setting. The question which automatically arises is if such states exist, i.e. if the set $\mathfrak{S}_p(\mathfrak{A})$ is non-empty. In order to answer that question, we have to introduce the weak*-topology in the set \mathfrak{A}^* . For any finite set of operators $A_1, \dots, A_n \in \mathfrak{A}$, we can define a seminorm

σ_{A_1, \dots, A_n} on \mathfrak{A}^* by

$$\sigma_{A_1, \dots, A_n}(\phi) = \sup \{ |\phi(A_k)| : k = 1, \dots, n \} . \quad (2.20)$$

The weak*-topology in \mathfrak{A}^* is defined as the topology generated by the open neighborhoods

$$\mathcal{U}(\phi; A_1, \dots, A_n; \epsilon) = \{ \phi' \in \mathfrak{A}^* : \sigma_{A_1, \dots, A_n}(\phi - \phi') < \epsilon \} , \quad (2.21)$$

for all $\phi \in \mathfrak{A}^*$, $A_1, \dots, A_n \in \mathfrak{A}$ and $\epsilon > 0$. In particular, a sequence $\phi_k \in \mathfrak{A}^*$ ($k \in \mathbb{N}$) converges to $\phi \in \mathfrak{A}^*$ in the weak*-topology iff

$$\phi_k(A) \xrightarrow[k \rightarrow \infty]{} \phi(A) , \quad \text{for all } A \in \mathfrak{A} . \quad (2.22)$$

Lemma 2.24. *Let \mathfrak{A} be a C^* -algebra.¹² Then the set of pure states $\mathfrak{S}_p(\mathfrak{A})$ is non-empty, and any state in $\mathfrak{S}(\mathfrak{A})$ is the weak*-limit of some sequence of finite convex combinations of pure states.*

Proof. See [2]. □

In other words, the set of pure states $\mathfrak{S}_p(\mathfrak{A})$ “generates” the whole set of states $\mathfrak{S}(\mathfrak{A})$, if we allow infinite convex combinations converging in the weak*-topology.

The weak*-topology has also a physical meaning. Suppose that we prepare our physical system in some physical state represented by the state $\omega \in \mathfrak{S}(\mathfrak{A})$. Since in physics, we can never perform infinitely many experiments and since each experiment has a limited accuracy we can, by monitoring the state ω , never determine more than some weak*-neighborhood in $\mathfrak{S}(\mathfrak{A})$ of the form (2.21). In other words, the state ω is “physically indistinguishable” from any other state belonging to such a weak*-neighborhood, or equivalently, any other state belonging to such a weak*-neighborhood may represent the same physical state than ω . We will return to this issue when we discuss the notion of physical equivalence between representations.

2.1.3. Representations

In general, one chooses an appropriate representation of the abstract algebra \mathfrak{A} , as bounded operators in some Hilbert space, to better describe the physical system.

Definition 2.25. Let \mathfrak{A} be a C^* -algebra. A *representation* π is a morphism $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ where \mathcal{H}_π is some Hilbert space depending on the representation. π is called the *physical representation* and \mathcal{H}_π the *physical Hilbert space*.

¹²For this lemma, the algebra must be unital, i.e. it contains an identity operator.

The following lemma asserts that any representation is continuous, and the image of algebra of observables \mathfrak{A} under any representation is indeed a concrete C^* -algebra.

Lemma 2.26. *Let \mathfrak{A} be a C^* -algebra and $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ a representation. Then, π is an homomorphism and $\pi(\mathfrak{A})$ is a C^* -subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$.*

Proof. See [21]. □

If our physical system has finitely many coordinates and momenta which satisfy the canonical commutation relations (example 2.15), the choice of representation is uniquely determined (up to unitary equivalence) by the Stone-von Neumann theorem.

Theorem 2.27. *(Stone-von Neumann) Let \mathcal{W}_n the Weyl algebra of example 2.15. Then there exists a unique (up to unitary equivalence) irreducible representation $\pi : \mathcal{W}_n \rightarrow L^2(\mathbb{R}^n)$, called the Schrodinger's representation. It is given by*

$$\pi(W_q(\bar{a}))\psi(\bar{x}) = e^{i\bar{x}\cdot\bar{a}}\psi(\bar{x}) \quad \text{and} \quad \pi(W_p(\bar{b}))\psi(\bar{x}) = \psi(\bar{x} + \bar{b}). \quad (2.23)$$

If we define $\pi(W_q(\bar{a})) := e^{i\bar{q}\cdot\bar{a}}$ and $\pi(W_p(\bar{b})) := e^{i\bar{p}\cdot\bar{b}}$, then we obtain the well-known expressions

$$q_j\psi(\bar{x}) = x_j\psi(\bar{x}) \quad \text{and} \quad p_j\psi(\bar{x}) = -i\frac{\partial}{\partial x_j}\psi(\bar{x}). \quad (2.24)$$

Furthermore, any reducible representation is a copy of Schrodinger's representation.

Proof. See [52]. □

However, if the system has infinitely many degrees of freedom, as in QFT, the Stone-von Neumann theorem is no longer applicable, and in such cases, there are many unitarily inequivalent representations of the canonical commutation relations [1]. Therefore, for such systems the physical representation π should be chosen carefully, depending on the particular dynamics of the system at hand; for instance, the Fock representation cannot be used for interacting fields. Without loss of generality, we may always assume that the physical representation π is *faithful*, i.e. that π is injective. The reason for this is as follows. Suppose that it would have been possible to physically describe a quantum system by using a non-faithful representation π of the algebra of observables \mathfrak{A} . Then the representation π defines a faithful representation $\tilde{\pi}$ of the quotient $\tilde{\mathfrak{A}} := \mathfrak{A}/\ker(\pi)$, and we could just as well have started with this quotient algebra (as the algebra of observables) from the beginning.

Once a representation is specified, we can identify a preferred subset of the set of all states.

Definition 2.28. Let \mathfrak{A} be a C^* -algebra and $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ a representation. A state $\omega \in \mathfrak{S}(\mathfrak{A})$ is said to be *normal with respect to the representation π* or *π -normal* if there

exists a density matrix ρ_ω such that $\omega(\cdot) = \text{Tr}(\rho_\omega \pi(\cdot))$. The set of all π -normal states is a convex subset of the set of all states $\mathfrak{S}(\mathfrak{A})$. It is called the *folium* of π and it is denoted by $\mathfrak{S}^{(\pi)}(\mathfrak{A})$. Furthermore, any normal state of the form $\omega(\cdot) = \text{Tr}(|\Psi\rangle\langle\Psi| \pi(\cdot))$, with unit vector $|\Psi\rangle \in \mathcal{H}_\pi$, is called a *vector state of the representation* π .

The set of all π -normal states is the subset of the set of all states that, in some way, have a simpler description in the representation π . In the rest of this subsection, we explain that the choice of the physical representation is a sort of mathematical convenience. In other words, if we want to study some physical processes involving a particular state, the best idea is to pick up a representation where such a state is represented by a density matrix, or even better, is represented by a vector state.¹³ Indeed, it can be defined a notion of “physical equivalence” between representations, that relates representations whose set of normal states can not be distinguished performing finitely many experiments with limited accuracy. In the end, it can be proven, that all faithful representations of the algebra of observables are physically equivalent.

The following theorem asserts that any general state of $\mathfrak{S}(\mathfrak{A})$ can be approximated by π -normal states.

Lemma 2.29. *The set of normal states $\mathfrak{S}^{(\pi)}(\mathfrak{A})$ of any faithful representation π is weakly*-dense in the set of all states $\mathfrak{S}(\mathfrak{A})$.*

Proof. See [2]. □

The following definition concerns two different notions of equivalence between representations.

Definition 2.30. Let \mathfrak{A} be a C^* -algebra, and $\pi_1 : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_{\pi_1})$ and $\pi_2 : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_{\pi_2})$ two representations. We say that

- π_1 is *unitarily equivalent* to π_2 ($\pi_1 \cong \pi_2$) if exists a unitary operator $U : \mathcal{H}_{\pi_1} \rightarrow \mathcal{H}_{\pi_2}$ such that $U\pi_1(A)U^* = \pi_2(A)$ for all $A \in \mathfrak{A}$,
- and π_1 is *phenomenologically equivalent*¹⁴ to π_2 if $\mathfrak{S}^{(\pi_1)}(\mathfrak{A}) = \mathfrak{S}^{(\pi_2)}(\mathfrak{A})$.

The notion of unitary equivalence says that both representations are essentially equal from the mathematical point of view, whereas the notion of phenomenological equivalence asserts that both representations are good for the study of the same subset of states. Of course, any two unitarily equivalent representations are also phenomenological equivalent, but the converse is false. The third notion of equivalence that we introduce is physical equivalence.

¹³It is always possible. See theorem (2.34).

¹⁴In the literature, *quasi-equivalent* is used as a synonym of phenomenological equivalent.

Definition 2.31. Let \mathfrak{A} be a C^* -algebra and $\pi_1 : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_{\pi_1})$ and $\pi_2 : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_{\pi_2})$ two representations. We say that π_1 is *physically equivalent* to π_2 if for every state $\omega_1 \in \mathfrak{S}^{(\pi_1)}(\mathfrak{A})$ and every weak*-neighborhood $\mathcal{U}(\omega_1; A_1, \dots, A_n; \epsilon)$ of ω_1 there exists a state $\omega_2 \in \mathfrak{S}^{(\pi_2)}(\mathfrak{A})$ such that $\omega_2 \in \mathcal{U}(\omega_1; A_1, \dots, A_n; \epsilon)$.

Remark 2.32. The definition of physical equivalence does not look like symmetric between π_1 and π_2 , however, it can be shown that it is indeed an equivalence relation.

The physical significance of the above definition comes from the following mental experiment. One starts with a physical state¹⁵ Ω that is represented by the element $\omega_1 \in \mathfrak{S}^{(\pi_1)}(\mathfrak{A})$ belonging to the folium of some representation π_1 . But, how could we ensure that ω_1 indeed represents the physical state Ω ? What we do in a real experiment, is to take a collection of observables represented by selfadjoint operators $A_1, \dots, A_n \in \mathfrak{A}$ and measure such observables in our physical state Ω , to obtain the results $a_1, \dots, a_n \in \mathbb{R}$.¹⁶ Since we can only measure a finite number of observables, and since each observation has a finite precision, the claim that our state ω_1 represents the physical state Ω , actually means that

$$|\omega_1(A_k) - a_k| < \epsilon_k, \quad \text{for all } k = 1, \dots, n, \quad (2.25)$$

where $\epsilon_k > 0$ represents the accuracy of such measurements. Equation (2.25) tells us that any real experiment does not determine the mathematical state ω_1 uniquely, but instead a weak*-neighborhood of it. Hence, for theoretical computations, any other mathematical state ω_2 belonging to such a weak*-neighborhood represents the same physical state Ω . Two representations π_1 and π_2 are physical equivalent if the folium of π_1 is indistinguishable from the folium of π_2 after performing any real experiment. The foregoing discussion culminates in the following theorem.

Theorem 2.33. *Any two faithful representations of the algebra of observables are physically equivalent.*

Proof. See [53]. □

In other words, all faithful representations describe the same physics. The selection of a particular representation is a matter of convenience without physical implications. If one wants to study some process related to some particular state $\omega \in \mathfrak{S}(\mathfrak{A})$, it is better to choose a representation where such a state is normal, i.e. is described by a density matrix. As a general conclusion, the physics is encoded in the algebra of observables \mathfrak{A} itself.¹⁷

¹⁵Here Ω does not represent any object in any mathematical space. It is just a way to call the real physical state prepared in our laboratory.

¹⁶We have to repeat the experiment many times in order to pursue all such measurements.

¹⁷And of course, in the particular chosen state.

2.1.4. Relation between states and representations

The following theorem not only ensures that representations already exist, but also that any state $\omega \in \mathfrak{S}(\mathfrak{A})$ can be viewed as a vector state in some representation.¹⁸

Theorem 2.34. (*GNS construction*) *Let \mathfrak{A} be a C^* -algebra and $\omega \in \mathfrak{S}(\mathfrak{A})$ a state. There exists a unique (up to unitarily equivalence) representation $\pi_\omega : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ with cyclic¹⁹ vector $|\Omega\rangle \in \mathcal{H}_\omega$ such that*

$$\omega(A) = \langle \Omega | \pi_\omega(A) | \Omega \rangle \quad \text{for all } A \in \mathfrak{A}. \quad (2.26)$$

The above representation is usually called as the GNS-representation of the state ω .

Proof. Here we give an sketch of the proof. For a complete mathematical proof see [21]. The idea is to start defining the pre-Hilbert space $\mathcal{H}_0 \equiv \mathfrak{A}$ as a set, with inner product given by $\langle A | B \rangle_0 = \omega(A^*B)$. Then we define the set²⁰ of null vectors $\mathcal{N} := \{A \in \mathcal{H}_0 : \omega(A^*A) = 0\}$. The Hilbert space \mathcal{H}_ω is defined as the completion of the quotient $\mathcal{H}_0/\mathcal{N}$, where the completion is in the norm derived from $\langle \cdot | \cdot \rangle_0$. A dense subset of vectors in \mathcal{H}_ω is given by the equivalence class of elements in $\mathcal{H}_0 = \mathfrak{A}$, which are simply denoted by $|A\rangle$ with $A \in \mathfrak{A}$. The inner product in \mathcal{H}_ω is the one derived from \mathcal{H}_0 , i.e. $\langle A | B \rangle = \omega(A^*B)$. The representation is defined by $\pi_\omega(A) |B\rangle = |AB\rangle$ over the elements in $\mathcal{H}_0/\mathcal{N}$ and extended by continuity to the whole \mathcal{H}_ω . The unit vector $|\Omega\rangle$ is represented (up to a phase) by $|\mathbf{1}\rangle$, where $\mathbf{1} \in \mathfrak{A}$ is the identity operator. \square

The above theorem gives us the first relation between states and representations. The following theorem shows that the notions of purity of the state and irreducibility of its GNS-representation are intimately related.

Theorem 2.35. *Let \mathfrak{A} a C^* -algebra, $\omega \in \mathfrak{S}(\mathfrak{A})$ a state and $\pi_\omega : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ its GNS-representation. Then π_ω is irreducible if and only if ω is pure.²¹*

Proof. See [21]. \square

There is a second relation between states and representations, based on the fact that the set of pure states $\mathfrak{S}_p(\mathfrak{A})$ is decomposed as a disjoint union of subsets called coherent sets or sectors. For that, we define the *transition probability* $\omega_1 \cdot \omega_2$ between two pure states $\omega_1, \omega_2 \in \mathfrak{S}_p(\mathfrak{A})$ as

$$\omega_1 \cdot \omega_2 = 1 - \frac{1}{4} \|\omega_1 - \omega_2\|, \quad (2.27)$$

¹⁸This is the algebraic abstract version of purification, which is well-known for finite dimensional algebras or full operators algebras $\mathcal{B}(\mathcal{H})$.

¹⁹Cyclic vector means that $\pi_\omega(A) |\Omega\rangle$ is dense in \mathcal{H}_ω .

²⁰ \mathcal{N} is a left ideal of the algebra \mathfrak{A} .

²¹A representation $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ is said to be *irreducible* if \mathcal{H} has no proper closed $\pi(\mathfrak{A})$ -invariant subspaces. According to Schur's lemma, π is irreducible if and only if $\pi(\mathfrak{A})' = \{\lambda \cdot \mathbf{1}\}$.

where the norm is the one defined in (2.18).²² Because $0 \leq \|\omega_1 - \omega_2\| \leq \|\omega_1\| + \|\omega_2\|$, it is clear that $\omega_1 \cdot \omega_2 \in [0, 1]$, and it follows from the positive-definiteness of $\|\cdot\|$ that $\omega_1 \cdot \omega_2 = 1$ if and only if $\omega_1 = \omega_2$. When $\omega_1 \cdot \omega_2 = 0$, we say that the states ω_1 and ω_2 are *orthogonal*. Two subsets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathfrak{S}_p(\mathfrak{A})$ are called *mutually orthogonal* if $\omega_1 \cdot \omega_2 = 0$ for all $\omega_1 \in \mathcal{S}_1$ and $\omega_2 \in \mathcal{S}_2$, and it is denoted by $\mathcal{S}_1 \perp \mathcal{S}_2$. This motivates the following definition.

Definition 2.36. A non-empty subset $\mathcal{S} \subset \mathfrak{S}_p(\mathfrak{A})$ is called *indecomposable* iff it cannot be written as the union of two non-empty, mutually disjoint and mutually orthogonal subsets, i.e. $\exists \mathcal{S}_1, \mathcal{S}_2 \subset \mathfrak{S}_p(\mathfrak{A})$ such that $\mathcal{S}_1, \mathcal{S}_2 \neq \emptyset$, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, $\mathcal{S}_1 \perp \mathcal{S}_2$ and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$.

Proof. See [1]. □

With the help of the above definition, we define the following relation on $\mathfrak{S}_p(\mathfrak{A})$.

Definition 2.37. Let $\omega_1, \omega_2 \in \mathfrak{S}_p(\mathfrak{A})$. We define the relation $\omega_1 \sim \omega_2$ if there exists an indecomposable set $\mathcal{S} \subset \mathfrak{S}_p(\mathfrak{A})$ such that $\omega_1, \omega_2 \in \mathcal{S}$.

Lemma 2.38. *The relation $\omega_1 \sim \omega_2$ on $\mathfrak{S}_p(\mathfrak{A})$ is an equivalence relation. Then the set of pure states can be uniquely partitioned into non-empty sets (mutually disjoint and mutually orthogonal), which are precisely the equivalence classes in $\mathfrak{S}_p(\mathfrak{A})$. Furthermore, each equivalence class is an indecomposable set.*

Proof. See [1]. □

Each equivalence class in above lemma is called a (*superselection*) *sector*. We usually write $\mathfrak{S}_p(\mathfrak{A}) = \bigcup_{\alpha \in \mathcal{I}}^{(d)} \mathcal{C}_\alpha$, where $\{\mathcal{C}_\alpha\}_{\alpha \in \mathcal{I}}$ are the sectors and (d) denotes that union is along non-empty, mutually disjoint ($\mathcal{C}_\alpha \cap \mathcal{C}_\beta = \emptyset$ if $\alpha \neq \beta$) and mutually orthogonal ($\mathcal{C}_\alpha \perp \mathcal{C}_\beta = \emptyset$ if $\alpha \neq \beta$) sets. The following example helps us to better understand the above definitions.

Example 2.39. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and define $\mathfrak{A} := \mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$. Any operator $A \in \mathfrak{A}$ can be uniquely represented as $A = (A_1, A_2)$, with $A_k \in \mathcal{B}(\mathcal{H}_k)$ for $k = 1, 2$. The algebraic operations in \mathfrak{A} are component-wise, and within each component are just the usual operations in $\mathcal{B}(\mathcal{H}_k)$. Following definition 2.21 we find that the set of pure states $\mathfrak{S}_p(\mathfrak{A})$ is in one-to-one correspondence with the subset of vectors $\mathcal{H}_1 \cup \mathcal{H}_2$, where \cup denotes the union of sets, and not the direct sum. In fact, any state given by a vector of the form $\alpha_1 |\Psi_1\rangle + \alpha_2 |\Psi_2\rangle$, with $|\Psi_k\rangle \in \mathcal{H}_k$ and $\alpha_k \neq 0$, is a mixed state for \mathfrak{A} . It correspond to the state $|\alpha_1|^2 \omega_1 + |\alpha_2|^2 \omega_2$, where $\omega_i(\cdot) = \text{Tr}(|\Psi_i\rangle \langle \Psi_i| \cdot)$. The set of pure states $\mathfrak{S}_p(\mathfrak{A})$ is not indecomposable since the subsets of pure states represented by \mathcal{H}_1 and \mathcal{H}_2 gives a non-trivial decomposition of $\mathfrak{S}_p(\mathfrak{A})$. Furthermore, the subsets of pure states represented by \mathcal{H}_1 and \mathcal{H}_2 are themselves indecomposable. Hence, the sectors are in one-to-one correspondence with the subspaces $\{\mathcal{H}_1, \mathcal{H}_2\}$.

²²Equation (2.27) coincides with the usual expression $\omega_1 \cdot \omega_2 = |\langle \Psi_1 | \Psi_2 \rangle|^2$ when $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ and the pure states are defined by unit vectors $|\Psi_i\rangle \in \mathcal{H}$, i.e. $\omega_i(\cdot) = \text{Tr}(|\Psi_i\rangle \langle \Psi_i| \cdot)$.

There is a close relation between sectors and irreducible representations. It is summarized in the following theorem.

Theorem 2.40. *Each superselection sector of $\mathfrak{S}_p(\mathfrak{A})$ coincides with the set of vector states in some irreducible representation. Conversely, the set of vector states of an irreducible representation coincides with some superselection sector of $\mathfrak{S}_p(\mathfrak{A})$. Moreover, sets of vector states of unitarily equivalent representations corresponds to the same sector.*

Proof. See [1]. □

The above theorem gives the famous one-to-one correspondence between sectors of pure states and irreducible representations. Since any other state of $\mathfrak{S}(\mathfrak{A})$ is a weak*-limit of convex combinations of pure states, then for computational purposes, it is enough to consider only unitarily inequivalent irreducible representations.

2.1.5. von Neumann algebras

Among the class of concrete C^* -algebras,²³ there is the subclass of von Neumann algebras. To properly define it, we have to introduce the *weak operator topology* in $\mathcal{B}(\mathcal{H})$. It is defined by the set of seminorms $\varrho_{\Psi, \Phi} : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{R}_{\geq 0}$,

$$\varrho_{\Psi, \Phi}(A) := |\langle \Psi | A | \Phi \rangle|, \quad (2.28)$$

where $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}$. In particular, a sequence $A_k \in \mathcal{B}(\mathcal{H})$ ($k \in \mathbb{N}$) converges to $A \in \mathcal{B}(\mathcal{H})$ in the weak operator topology iff

$$\langle \Psi | A_k | \Phi \rangle \xrightarrow[k \rightarrow \infty]{} \langle \Psi | A | \Phi \rangle, \quad \text{for all } |\Psi\rangle, |\Phi\rangle \in \mathcal{H}. \quad (2.29)$$

Definition 2.41. Any $*$ -subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ that contains the identity operator $\mathbf{1}$ and is closed in the weak operator topology is called a *von Neumann (vN) algebra*.

Since the weak operator topology is weaker than the uniform topology, we have that any vN algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is also a concrete C^* -algebra. The converse is false.²⁴ In fact, given any C^* -algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ we can define the *vN algebra generated* by \mathfrak{A} simply as its closure in the weak operator topology, i.e. including in the algebra \mathfrak{A} the limit points of weak convergent sequences.

Remark 2.42. Besides the uniform topology (see definition 2.6) and the weak operator topology (see equations (2.28-2.29)) there are other five natural topologies in $\mathcal{B}(\mathcal{H})$: σ -weak, σ -weak*, strong, σ -strong, σ -strong*. The above seven topologies are inequivalent

²³Remember that due to theorem 2.13, we do not lose generality if we only consider concrete C^* -algebras.

²⁴For finite-dimensional algebras, both notions coincide.

over $\mathcal{B}(\mathcal{H})$, but a $*$ -subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is closed in the weak operator topology iff is closed in any of the other five topologies (excluding the uniform topology). In other words, in definition 2.41, we can replace the closeness in the weak operator topology with any of the other five topologies (except the uniform topology). See [21].

Remark 2.43. vN algebras could be defined in an abstract sense without referring to any Hilbert space. Such objects are known as W^* -algebras. Fortunately, there is a theorem that asserts that any W^* -algebra is equivalent to a vN subalgebra of $\mathcal{B}(\mathcal{H})$. See [54].

The famous von Neumann bicommutant theorem gives a pure algebraic characterization for vN algebras. To state it, we have first to introduce the following definition.

Definition 2.44. Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$. We define its *commutant* as

$$\mathcal{S}' := \{A \in \mathcal{B}(\mathcal{H}) : [A, B] = 0 \text{ for all } B \in \mathcal{S}\}, \quad (2.30)$$

where $[A, B] = AB - BA$ denotes the *commutator*. The *bicommutant* of \mathcal{S} is simply defined as $\mathcal{S}'' := (\mathcal{S}')'$. Of course, we have that $\mathcal{S} \subset \mathcal{S}''$.

Theorem 2.45. (*von Neumann bicommutant theorem*) Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint set.²⁵ Then \mathcal{S}' is a vN algebra and \mathcal{S}'' is the smallest vN algebra containing \mathcal{S} . Furthermore, $(\mathcal{S}'')' = \mathcal{S}'$. In particular, if $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ is a C^* -algebra, then \mathfrak{A}'' is the closure of \mathfrak{A} in the weak operator topology, i.e. \mathfrak{A}'' is the vN algebra generated by \mathfrak{A} .

Proof. See [21]. □

Corollary 2.46. $\mathcal{B}(\mathcal{H})$ and $\mathbf{1} := \{\lambda \mathbf{1} : \lambda \in \mathbb{C}\}$ are vN algebras.

The above theorem is very useful for practical purposes. Suppose that we have a quantum system represented by $\mathcal{B}(\mathcal{H})$, but we have at our disposal some subset of observables represented by the self-adjoint operators $\{A_1, A_2, \dots\}$. Then, $\{A_1, A_2, \dots\}''$ is the smallest vN algebra generated by such a set of operators.

Now we introduce some useful definitions.

Definition 2.47. The *center* of a vN algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is defined as $\mathcal{Z}(\mathcal{A}) := \mathcal{A} \cap \mathcal{A}'$, which is also a vN algebra. If the center is trivial, i.e. $\mathcal{Z}(\mathcal{A}) = \mathbf{1}$, we say that \mathcal{A} is a *factor*.

Definition 2.48. Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{B}(\mathcal{H})$ be vN algebras. If $\mathcal{A}_1 \subset \mathcal{A}_2$ we say that \mathcal{A}_1 is a *vN subalgebra* of \mathcal{A}_2 . In particular, \mathcal{A}_1 and \mathcal{A}_2 are vN subalgebras of $\mathcal{B}(\mathcal{H})$.

Definition 2.49. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra and $|\Psi\rangle \in \mathcal{H}$ a vector. We say that $|\Psi\rangle$ is *cyclic* for \mathcal{A} if $\mathcal{A}|\Psi\rangle$ is dense in \mathcal{H} , and we say that $|\Psi\rangle$ is *separating* for \mathcal{A} if $\bar{\mathcal{A}}A \in \mathcal{A}$ such that $A|\Psi\rangle = 0$. Any vector which is cyclic and separating is called *standard*.

²⁵ $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ is called a self-adjoint set if $A \in \mathcal{S} \Rightarrow A^* \in \mathcal{S}$.

Lemma 2.50. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra and $|\Psi\rangle \in \mathcal{H}$ a vector. Then, $|\Psi\rangle$ is cyclic (rep. separating) for \mathcal{A} if and only if $|\Psi\rangle$ is separating (rep. cyclic) for \mathcal{A}' .*

Proof. See [21]. □

vN algebras can be thought of as abstract algebras represented in some Hilbert space. In fact, any representation $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ of some C^* -algebra \mathfrak{A} , gives place naturally to the vN algebra $\pi(\mathfrak{A})''$. A vN algebra has naturally a preferred set of normal states, i.e. all the states represented by density matrices in the Hilbert space where the vN algebra acts.

Despite that there is a very clear mathematical distinction between C^* -algebras and vN algebras, there is no general consensus if the algebras of observables of quantum systems may be described by uniform or weakly closed algebras. Both type of algebras are used in the literature, and sometimes simultaneously in the description of a system. Below we give some partial arguments in favor of vN algebras with respect to C^* -algebras. We also do not have to forget that vN algebras are also C^* -algebras, and hence, all the discussion we have done about C^* -algebras also applies to vN algebras.

1. When we construct examples of algebras of observables (in particular in algebraic quantum field theory), we first define some set of observables through concrete mathematical expressions, and then, we define the algebra of observables as the closure of such a set. From the technical point of view, it is easier to take a weak closure than the uniform closure. It is because the weak closure is equivalent to the double commutant, which is an algebraic operation.
2. In quantum information theory, most of the information/statistical/entanglement measures have easier definitions for vN algebras rather than for C^* -algebras. In fact, some measures over C^* -algebras are simply defined in term of the vN algebra generated of some particular representation (e.g. the GNS-representation of some state).
3. In QFT, in the DHR theory of superselection sectors (see chapter 7), the local algebras corresponding to bounded regions are defined to be vN algebras in order to have a simpler description of the structure of superselection sectors. However, the algebras associated with unbounded regions are defined to be C^* -algebras but not vN algebras.
4. In QFT, it is well-known that operators belonging to spacelike separated regions commute. In fact, for any spacetime region \mathcal{O}_1 we have an associated algebra of observables $\mathfrak{A}(\mathcal{O}_1)$.²⁶ Suppose for a moment that all local algebras are subalgebras of a concrete algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$. Given any region \mathcal{O}_2 spacelike separated to \mathcal{O}_1 , we must have $\mathfrak{A}(\mathcal{O}_2) \subset \mathfrak{A}(\mathcal{O}_1)'$. $\mathfrak{A}(\mathcal{O}_1)$ may not be a vN algebra, but $\mathfrak{A}(\mathcal{O}_1)'$ is always

²⁶We make such a correspondence clearer in the following section.

a vN algebra. vN algebras arise naturally in QFT when we impose the principle of microcausality.

Finally, we want to remark that the set of vN subfactors of $\mathcal{B}(\mathcal{H})$ has a natural structure of orthocomplemented lattice.²⁷

Proposition 2.51. *The set of vN subfactors of $\mathcal{B}(\mathcal{H})$ has a natural structure of orthocomplemented lattice (ordered by inclusion) given by*

$$\{\lambda\mathbf{1}\} \subset \mathcal{A} \subset \mathcal{B}(\mathcal{H}), \quad (2.31)$$

$$\mathcal{A}_1 \vee \mathcal{A}_2 = (\mathcal{A}_1 \cup \mathcal{A}_2)'' , \quad (2.32)$$

$$\mathcal{A}_1 \wedge \mathcal{A}_2 = \mathcal{A}_1 \cap \mathcal{A}_2 , \quad (2.33)$$

where $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{B}(\mathcal{H})$ are any vN subalgebras and \cup (resp. \cap) denote the union (resp. intersection) of sets. The complementation in the lattice is given by the commutant $\mathcal{A} \rightarrow \mathcal{A}'$.

Proof. See [2]. □

Such a structure plays an important role in the correspondence between algebras and regions in QFT since the set of causally complete regions of the spacetime has also a natural structure of orthocomplemented lattice. We will extend this discussion in subsection 2.2.5.

2.2. Algebraic quantum field theory

In the algebraic approach to quantum field theory (AQFT), we associate for each region of the spacetime a C^* or vN algebra which encodes the algebraic relations between the quantum fields. Such an assignment must satisfy a set of axioms that encode the physical conditions in the algebraic framework. Unless the specific set of axioms considered could depend on the underlying theory (especially on the spacetime considered), the minimal assumptions we list below are very standard for the treatment of QFT on Minkowski spacetime.

2.2.1. Axioms of AQFT

To start, we define the d -dimensional Minkowski spacetime as the smooth manifold \mathbb{R}^d with Lorentzian metric $\eta_{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$. The (proper orthochronous) Poincaré group is denoted by \mathcal{P}_+^\uparrow , and a general element $g \in \mathcal{P}_+^\uparrow$ is denoted by $g = (\Lambda, a)$, where $\Lambda \in \mathcal{L}_+^\uparrow$ is a (proper orthochronous) Lorentz matrix and $a \in \mathbb{R}^d$ is a d -vector which

²⁷See appendix A for the definition of orthocomplemented lattice.

implements the spacetime translations. \mathcal{P}_+^\uparrow acts geometrically over any spacetime region $\mathcal{O} \subset \mathbb{R}^d$ as

$$g\mathcal{O} := \{\Lambda x + a : x \in \mathcal{O}\} . \quad (2.34)$$

Among the class of all spacetime regions, we work with the subclass of causally complete regions.

Definition 2.52. Given an $\mathcal{O} \subset \mathbb{R}^d$, we define its (open) *spacelike complement* \mathcal{O}' as the set of all points in \mathbb{R}^d that are spacelike separated with all points in \mathcal{O} , i.e.

$$\mathcal{O}' := \text{int} \{x \in \mathbb{R}^d : (x - y)^2 < 0, \forall y \in \mathcal{O}\} , \quad (2.35)$$

and its *causal completion* as $\mathcal{O}'' := (\mathcal{O}')'$. It is always true that $(\mathcal{O}'')' = \mathcal{O}'$. We say that a region \mathcal{O} is *causally complete* if $\mathcal{O}'' = \mathcal{O}$.²⁸ The set of all causally complete regions is denoted by \mathcal{K} .

A useful way to construct causally complete regions is as follows. First, define a (smooth) Cauchy surface $\Sigma \subset \mathbb{R}^d$ and take any relative open subset $\mathcal{C} \subset \Sigma$. Then, its Cauchy development²⁹ $D(\mathcal{C})$ is a causally complete region. Furthermore, any causally complete region can be constructed in that way. We also have that \mathcal{P}_+^\uparrow maps the set of causally complete regions onto the set of causally complete regions, i.e. $g(\mathcal{K}) = \mathcal{K}$ for all $g \in \mathcal{P}_+^\uparrow$.

There is a particular kind of causally complete regions, known as double cones.

Definition 2.53. We call a *double cone* to any non-empty open region $\mathcal{O} \subset \mathbb{R}^d$ of Minkowski spacetime defined by the intersection of the future open null cone of some point $x \in \mathbb{R}^d$ with the past open null cone of another point $y \in \mathbb{R}^d$.³⁰

Remark 2.54. Every double cone is a bounded, connected and simply connected region. Moreover, \mathcal{P}_+^\uparrow maps the set of double cones onto the set of double cones.

Before we list the axioms of AQFT, we define two different notions of spacelike separated regions.

Definition 2.55. We say that $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ are *spacelike separated* if $\mathcal{O}_1 \subset \mathcal{O}'_2$, and *strictly spacelike separated* if $\overline{\mathcal{O}}_1 \subset \mathcal{O}'_2$, where $\overline{\mathcal{O}}_1$ denotes the closure of \mathcal{O}_1 . Spacelike and strictly spacelike separated regions are respectively denoted by $\mathcal{O}_1 \times \mathcal{O}_2$ and $\mathcal{O}_1 \times\!\!\!\times \mathcal{O}_2$. strictly spacelike separated regions are spacelike separated regions such that there is a finite non-zero separation distance between them.

²⁸A causally complete region is always an open set of \mathbb{R}^d .

²⁹Given any set $\mathcal{S} \subset \mathbb{R}^d$, its *future* (resp. *past*) *Cauchy development* $D^+(\mathcal{S})$ (resp. $D^-(\mathcal{S})$) is defined as the set of all spacetime points $x \in \mathbb{R}^d$ for which every past (resp. future) directed inextendible causal curve through x intersects \mathcal{S} at least once. The *Cauchy development* $D(\mathcal{S})$ is the union of the future and past Cauchy developments.

³⁰In particular, y must be in the timelike future of x in order to have $\mathcal{O} \neq \emptyset$.

Definition 2.56. An *algebraic quantum field theory (AQFT)* is defined by a C^* -algebra \mathfrak{A} , called the *algebra of observables*, and an assignment to every causally complete region $\mathcal{O} \in \mathcal{K}$ a C^* -subalgebra $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}$, i.e.

$$\mathcal{O} \in \mathcal{K} \mapsto \mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}, \quad (2.36)$$

which are called the *local algebras (of observables)*. This collection of local algebras must satisfy:

1. (*generating property*) $\mathfrak{A} = \overline{\bigcup_{\mathcal{O} \in \mathcal{K}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}$.
2. (*isotony*) For any pair of regions $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.
3. (*locality*) For any pair of regions $\mathcal{O}_1 \times \mathcal{O}_2$, then $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$.
4. (*Poincaré covariance*) There is a uniform continuous linear representation $\alpha : \mathcal{P}_+^\uparrow \rightarrow \text{Aut}(\mathfrak{A})$, such that $\alpha_g(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(g\mathcal{O})$ for all $\mathcal{O} \in \mathcal{K}$ and all $g \in \mathcal{P}_+^\uparrow$.

Any collection $\mathfrak{A}(\mathcal{O})$ of C^* -algebras satisfying the above axioms is called a *net of local algebras (of observables)*.³¹

Remark 2.57. In the usual approach to QFT, the theory is defined through a collection of field operators $\phi(x)$ acting on some Hilbert space \mathcal{H} . From a heuristic point of view, such field operators form a $*$ -algebra: we are allowed to take linear combinations of field operators, the product in the algebra is given by the composition and the involution is defined by taking adjoints. One is tempted to define the local algebras just as

$$\mathfrak{A}(\mathcal{O}) := \langle \{\phi(x) : x \in \mathcal{O}\} \rangle, \quad (2.37)$$

where $\langle \cdot \rangle$ means the algebra generated. Morally that is correct, but from the mathematical point of view, the field operators $\phi(x)$ are not operators in the usual sense (linear transformations acting on the Hilbert space). They are very singular objects known as (unbounded) operator-valued distributions. Luckily, such a technical problem can be bypassed, and a net of C^* -algebras can be properly defined from such a collection of fields (see subsection 2.2.3). Despite these technical issues, we want to emphasize that, relation (2.37) is an informal, but physically correct, way to define the local algebras.

There is an extra property that states that any local algebra may be generated by algebras of smaller regions.

5. (*additivity*) For any $\mathcal{O} \in \mathcal{K}$ and any of its open coverings $\mathcal{O} = \bigcup_\alpha \mathcal{O}_\alpha$ with $\mathcal{O}_\alpha \in \mathcal{K}$, then $\mathfrak{A}(\mathcal{O}) = \overline{\bigcup_\alpha \mathfrak{A}(\mathcal{O}_\alpha)}^{\|\cdot\|}$.

³¹Mathematically, due to axiom 1, the collection of local algebras form a net indexed by the elements of \mathcal{K} . The elements of \mathcal{K} form a directed set when it is ordered by the usual set inclusion.

This property may hold in physical models, but it is excluded from the axioms because there are consistent models of AQFT where additivity fails. We will discuss more about this fact in chapter 7, when we study theories having non-trivial superselection sectors.

One of the challenges of AQFT, is to find non-trivial models satisfying the axioms 1. to 4. above. However, this task is very difficult, and the number of known examples is very limited. Between them we want to emphasize the free scalar fields (see chapter 5 and [55]), free Dirac fields (see [55]) and chiral CFTs (see chapter 6 and [55, 56]).

2.2.2. Vacuum states and vacuum representations

In the usual approach to QFT, it is postulated the existence of a distinguished state, the vacuum state, which has the following characteristic properties: it is pure and Poincaré invariant. This motivates the following definition.

Definition 2.58. Let $\mathfrak{A}(\mathcal{O})$ be a net of local algebras and $\omega_0 \in \mathfrak{S}(\mathfrak{A})$ a state. We say that ω_0 is a *vacuum state* if it satisfies the above three conditions:

- ω_0 is pure, i.e. $\omega_0 \in \mathfrak{S}_p(\mathfrak{A})$.
- ω_0 invariant under all α_g , i.e. $\omega_0 \circ \alpha_g = \omega_0$ for all $g \in \mathcal{P}_+^\uparrow$.
- In the GNS-representation $\pi_0 : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_0)$ of ω_0 , the linear representation $\alpha : \mathcal{P}_+^\uparrow \rightarrow \text{Aut}(\mathfrak{A})$ is implemented by a positive energy unitary representation of $U : \mathcal{P}_+^\uparrow \rightarrow \mathcal{B}(\mathcal{H})$ in the sense that $U(g)\pi(A)U(g)^* = \pi(\alpha_g(A))$ for all $A \in \mathfrak{A}$ and all $g \in \mathcal{P}_+^\uparrow$. Positive energy means that the representation is strongly continuous and the infinitesimal generators P^μ of the translation subgroup (i.e. $U(0, a) = e^{iP^\mu a_\mu}$) have their joint spectrum included in the *closed forward light cone* $\bar{V}_+ := \{p \in \mathbb{R}^d : p \cdot p > 0 \text{ and } p^0 > 0\}$. Such a representation is called a *vacuum representation*.

Proposition 2.59. *In the context of the previous definition, the GNS-vector, which is denoted by $|0\rangle \in \mathcal{H}_0$, is called the vacuum vector, and it satisfies $U(g)|0\rangle = |0\rangle$ for all $g \in \mathcal{P}_+^\uparrow$.*

Remark 2.60. In the context of definition 2.58, a vacuum representation is always irreducible since the vacuum state is pure.

The question that arises is: given a net of local algebras satisfying the axioms of the definition 2.56, could one guarantee the existence and uniqueness of the vacuum state? Up to our knowledge, this question is an open problem. However, some insights on such problem can be found in [2, 57]. On the other hand, all known interesting AQFT examples

have a unique pure vacuum state. In this thesis, the existence and uniqueness of the vacuum are considered as extra postulates.

Once we have constructed the GNS-representation of the vacuum state, we can define the net of vN algebras

$$\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) := \pi_0(\mathfrak{A}(\mathcal{O}))'' \subset \mathcal{B}(\mathcal{H}_0). \quad (2.38)$$

It can be shown that such a net of vN algebras satisfies all the axioms of the definition 2.56. Moreover, under our assumptions, the algebra $\mathcal{A}(\mathbb{R}^d) = \mathcal{B}(\mathcal{H}_0)$ has a unique vacuum state given by the GNS-vector $|0\rangle$. For many purposes, it is very useful to consider this representation as being the defining net of our AQFT. In other words, the local algebras $\mathfrak{A}(\mathcal{O})$ are indeed concrete vN subalgebras of $\mathcal{B}(\mathcal{H}_0)$, such that the Poincaré automorphisms are unitarily implemented by positive energy unitary representation, and such that there is a unique Poincaré invariant vector $|0\rangle$ representing the vacuum state. For example, when one wants to construct a concrete model of an AQFT satisfying the axioms above, it is, in general, easier to construct a net of vN algebras $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$ acting on a Hilbert space.

Finally, we want to state the famous Reeh-Schlieder theorem. In order to do that, we have to introduce a weaker notion of additivity.

Definition 2.61. Let $\mathfrak{A}(\mathcal{O})$ be a net of local algebras. We say that the vacuum representation $\pi_0 : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_0)$ satisfies *weak additivity* if

$$\mathcal{B}(\mathcal{H}_0) = \left(\bigcup_{x \in \mathbb{R}^d} \pi_0(\mathfrak{A}(\mathcal{O} + x)) \right)'', \quad (2.39)$$

for all regions $\mathcal{O} \in \mathcal{K}$.

We emphasize that weak additivity holds in any interesting known model, and it is expected to hold in general. For the purpose of this thesis, weak additivity is considered as an extra axiom. Finally, we state the Reeh-Schlieder theorem.

Theorem 2.62. (*Reeh-Schlieder*) Let $\mathfrak{A}(\mathcal{O})$ be a net of local algebras which satisfies weak additivity. Then the vacuum vector $|0\rangle$ is cyclic for any algebra $\pi_0(\mathfrak{A}(\mathcal{O}))''$ of any non-empty region $\mathcal{O} \in \mathcal{K}$, and it is also separating for any algebra $\pi_0(\mathfrak{A}(\mathcal{O}))''$ of any region $\mathcal{O} \in \mathcal{K}$, whenever $\mathcal{O}' \neq \emptyset$.

Proof. See [58]. □

Definition 2.63. Motivated in the previous theorem, we say that a causally complete region $\mathcal{O} \in \mathcal{K}$ is *standard* whenever \mathcal{O} and \mathcal{O}' are non-empty.

2.2.3. AQFT vs. Garding-Wightman axioms

It is a very interesting exercise to try to relate the algebraic approach to other known approaches to QFT. A different axiomatic framework of QFT is the Garding-Wightman approach [41]. There, it is postulated the existence of point-like quantum fields $\phi(x)$, which are operator valued distributions in some Hilbert space \mathcal{H} .³² Operator valued distribution means that

$$\phi(f) := \int_{\mathbb{R}^d} \phi(x) f(x) d^d x, \quad (2.40)$$

is a well defined (unbounded) operator once it has been smeared by a test function $f \in \mathcal{S}(\mathbb{R}^d)$.³³ Since both approaches are mathematically well-defined, we can ask ourselves if they are equivalent (in some sense). This question can be split into two: 1) given a QFT theory satisfying the Garding-Wightman axioms, is it possible to construct an AQFT which encodes the same physical information? And conversely 2) given an AQFT, is it possible to construct a QFT satisfying the Garding-Wightman axioms?

The first question is well-understood, and the answer is affirmative if we supplement the Garding-Wightman axioms with some extra technical assumptions.³⁴ We sketch here such a construction for the real scalar field. It can be done in many different ways, but all of them giving the same result [59]. First of all, one could be tempted to define the local algebras for a region $\mathcal{O} \in \mathcal{K}$ as

$$\mathfrak{A}_U(\mathcal{O}) := \langle \{\phi(f) : \text{supp}(f) \subset \mathcal{O}\} \rangle, \quad (2.41)$$

where $\langle \cdot \rangle$ means the algebra generated. However, the operators $\phi(f)$ are unbounded, and hence, they cannot belong to the local algebras, which have to be defined including only bounded operators. Furthermore, an algebra containing unbounded operators can never be closed in order to obtain a C^* or a vN algebra. To bypass this issue, we have to turn from unbounded to bounded operators. Here, we sketch three different ways.

1. Since the operators $\phi(f)$ are self-adjoint (when f is a real function), due to the spectral theorem, they can be decomposed as

$$\phi(f) = \int_{\mathbb{R}} \lambda dE_{\phi(f)}(\lambda), \quad (2.42)$$

where $E_{\phi(f)}(\Delta)$, with $\Delta \subset \mathbb{R}$ a measurable set, are its spectral measures. For bounded measurable sets $\Delta \subset \mathbb{R}$, $E_{\phi(f)}(\Delta)$ are bounded operators. Then, the local

³²Along this section, we just consider a single real scalar field.

³³ $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of test functions (see for example [41]).

³⁴For the real scalar field, the assumption is that the operators $\phi(f)$ are self-adjoint whenever f is a real function. See [51] for the distinction between self-adjoint and hermitian operators.

(vN) algebras are defined as

$$\mathfrak{A}_1(\mathcal{O}) := \{E_{\phi(f)}(\Delta) : f \text{ real, } \text{sup}(f) \subset \mathcal{O} \text{ and } \Delta \subset \mathbb{R} \text{ bounded measurable}\}'' .$$

2. Since the operators $\phi(f)$ are self-adjoint (when f is a real function), due to Stone's theorem we have that the operators $W(f) := e^{i\phi(f)}$ are unitaries and hence bounded [51]. Then, the local (vN) algebras are defined as

$$\mathfrak{A}_2(\mathcal{O}) := \{W(f) : f \text{ real, } \text{sup}(f) \subset \mathcal{O}\}'' . \quad (2.43)$$

3. We first define the $*$ -algebra of (unbounded) operators $\mathfrak{A}_U(\mathcal{O})$ as in equation (2.41). Then, we define its *weak commutant* as³⁵

$$\mathfrak{A}(\mathcal{O})^w := \{A \in \mathcal{B}(\mathcal{H}) : \langle X^* \Psi | A | \Phi \rangle = \langle \Psi | A X | \Phi \rangle, \forall X \in \mathfrak{A}_U(\mathcal{O}), \forall |\Psi\rangle, |\Phi\rangle \in \mathcal{D}\} . \quad (2.44)$$

And finally, we define the local (vN) algebras as $\mathfrak{A}_3(\mathcal{O}) := (\mathfrak{A}(\mathcal{O})^w)'$.

It can be shown that $\mathfrak{A}_1(\mathcal{O}) = \mathfrak{A}_2(\mathcal{O}) = \mathfrak{A}_3(\mathcal{O}) =: \mathfrak{A}(\mathcal{O})$, and the net $\mathfrak{A}(\mathcal{O})$ satisfies all the axioms of the definition 2.56, regarding the quantum field $\phi(x)$ satisfies the Garding-Wightman axioms [59].

Question 2) is much harder, and at first sight, we must argue that it has a negative answer. In fact, the axioms of the definition 2.56 do not guarantee the existence of a vacuum state. If we supplement such axioms with the existence and uniqueness of the vacuum state, we should have a priori all the “physical” ingredients to perform such a construction. However, it is a very difficult task to construct a single point-like quantum field $\phi(x)$, with the property that $\phi(f)$ “generates” all the operators of all local algebras $\mathfrak{A}(\mathcal{O})$, when f runs along the set of all test functions. Given some extra technical assumptions, point-like objects $\phi(x)$, having some of the properties we expect for a quantum field, can be found out in the algebra \mathfrak{A} [60]. However, many details about the properties of such field operators remain unclear, e.g., the question of whether the fields possess finite spin, or the existence of product expansions [61]. On the other hand, there is some consensus among the algebraic community, that the point-like quantum fields are just “parameterizations” or “basis” for the abstract AQFT's. We argue in favor of such a statement with the help of some examples.

- The theory of the free massive scalar field $\phi(x)$ may look different than the theory of its derivatives $\partial_\mu \phi(x)$. But due to the equation of motion, we can reconstruct $\phi = \frac{1}{m^2} \square \phi$ from $\partial_\mu \phi$ using algebraic relations.

³⁵ $\mathcal{D} \subset \mathcal{H}$ denotes the common invariant dense domain of all quantum fields [41].

- Less trivial, the free massive Dirac fermion in $3 + 1$ dimensions has four field components. In the chiral basis, the algebra generated by these four field components is the same as the algebra generated by the 2-dimensional spinor components of definite chirality. This is because the other two components of the field are related to the former due to the equation of motion.
- Still less trivial, there is a duality that relates the free Maxwell in $2 + 1$ dimensions with the theory of the derivatives of a free massless scalar field in the same spacetime dimension. Such a duality is given by $\partial_\mu \phi = \varepsilon_{\mu\nu\sigma} F^{\nu\sigma}$. Then, the local algebras obtained from both theories must be equivalent.

These examples show that there may exist many different ways to “parametrize” the same AQFT using different sets of quantum fields. To find a complete and non-redundant set of quantum fields describing some AQFT looks very similar to the problem of finding a basis for an abstract (infinite-dimensional) vector space.

2.2.4. Fermions in AQFT

AQFT was born as a theory of local observables, i.e. the theory is defined to contain only observable quantities. As we have emphasized in the axioms, operators belonging to spacelike separated regions always commute. This is in contradiction with the usual approach of QFT, where the existence of fermionic fields that anticommute at spacelike distances is allowed. If we want to introduce fermionic operators, we have to relax the locality axiom (axiom 3 in definition 2.56), and we must introduce the notion of graded locality and graded net of local algebras.

For this purpose, let us assume that the local algebras are concrete C^* -algebras $\mathfrak{F}(\mathcal{O})$ acting on some Hilbert space \mathcal{H} , i.e. we have an assignment

$$\mathcal{O} \in \mathcal{K} \mapsto \mathfrak{F}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}) . \quad (2.45)$$

The global algebra is denoted by \mathfrak{F} . We first relax condition 4. in definition 2.56, imposing that the symmetry group is the universal covering $\tilde{\mathcal{P}}_+^\uparrow$ of the Poincaré group \mathcal{P}_+^\uparrow . A general element $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow$ is denoted by $\tilde{g} = (\tilde{\Lambda}, a)$, where $\tilde{\Lambda} \in \tilde{\mathcal{L}}_+^\uparrow$ and $a \in \mathbb{R}^d$. We denote by $g \in \mathcal{P}_+^\uparrow$ the image of $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow$ under the covering map. We assume that the Poincaré symmetry is unitarily implemented in the Hilbert space \mathcal{H} , i.e. there exists a positive energy unitary representation $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow \mapsto U(\tilde{g}) \in \mathcal{B}(\mathcal{H})$ such that $U(\tilde{g}) \mathfrak{F}(\mathcal{O}) U(\tilde{g})^* = \mathfrak{F}(g\mathcal{O})$. We also assume that there is a unique (up to a phase) unit vector $|0\rangle \in \mathcal{H}$ such that $U(\tilde{g})|0\rangle = |0\rangle$ for all $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow$.

Definition 2.64. A \mathbb{Z}_2 -grading in the net of local algebra $\mathfrak{F}(\mathcal{O})$ is defined by an operator $\Gamma \in \mathcal{B}(\mathcal{H})$ such that $\Gamma = \Gamma^{-1} = \Gamma^*$, $\Gamma|0\rangle = |0\rangle$, $\Gamma \mathfrak{F}(\mathcal{O}) \Gamma = \mathfrak{F}(\mathcal{O})$ for all $\mathcal{O} \in \mathcal{K}$

and $[\Gamma, U(\tilde{g})] = 0$ for all $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow$. An operator $A \in \mathfrak{F}$ such that $\Gamma A \Gamma = \pm A$ is called *homogeneous*, indeed a *Bose* or *Fermi* operator according to the \pm alternative, or simply *even* or *odd* operator. We shall say that the degree ∂_A of the homogeneous operator $A \in \mathfrak{F}$ is 0 in the Bose case and 1 in the Fermi case. A net of local algebras with a \mathbb{Z}_2 -grading is called a \mathbb{Z}_2 -graded net of local algebras.

Given a grading we have the following facts.

Proposition 2.65. *Let $\mathfrak{F}(\mathcal{O})$ be a \mathbb{Z}_2 -graded net of local algebras. Then*

1. *Any operator $A \in \mathfrak{F}$ can be uniquely decomposed as a sum of a Bose operator and a Fermi operator. In fact, $A = A_+ + A_-$ with $\Gamma A_\pm \Gamma = \pm A_\pm$, where $A_\pm := \frac{A \pm \Gamma A \Gamma}{2}$.*
2. *If $A = A_+ + A_- \in \mathfrak{F}(\mathcal{O})$, then $A_+, A_- \in \mathfrak{F}(\mathcal{O})$.*

Once we have defined the grading and we have identified the Bose and Fermi operators, we introduce the graded commutator.

Definition 2.66. Let $\mathfrak{F}(\mathcal{O})$ be a \mathbb{Z}_2 -graded net of local algebras. We define the *graded commutator* as follows: for $A, B \in \mathfrak{F}$ homogeneous we set

$$[A, B]_\Gamma := AB - (-1)^{\partial_A \cdot \partial_B} BA, \quad (2.46)$$

and we extend it to more general operators by linearity.

With the help of the graded commutator we could state the axiom of graded locality which replace the axiom 3 in definition 2.56.

Definition 2.67. Let $\mathfrak{F}(\mathcal{O})$ be a \mathbb{Z}_2 -graded net of local algebras. We say that the local algebra satisfies *graded locality* if for any pair of regions $\mathcal{O}_1 \times \mathcal{O}_2$, we have that $[\mathfrak{F}(\mathcal{O}_1), \mathfrak{F}(\mathcal{O}_2)]_\Gamma = \{0\}$.

The above notion is equivalent to the notion of twisted locality very often used in the literature.

Definition 2.68. Let $\mathfrak{F}(\mathcal{O})$ be a \mathbb{Z}_2 -graded net of local algebras and let $Z := \frac{1-i\Gamma}{1-i}$. We defined the *twisted commutant* of a local algebra as $\mathfrak{F}(\mathcal{O})^{t'} := Z \mathfrak{F}(\mathcal{O})' Z^*$. We say that the net satisfies *twisted locality* if for any pair of regions $\mathcal{O}_1 \times \mathcal{O}_2$, we have that $\mathfrak{F}(\mathcal{O}_1) \subset \mathfrak{F}(\mathcal{O}_2)^{t'}$.

Lemma 2.69. *Let $\mathfrak{F}(\mathcal{O})$ be a \mathbb{Z}_2 -graded net of local algebras. It satisfies graded locality if and only if it satisfies twisted locality.*

Proof. See [62]. □

We can summarize the content of this section in the following definition, which are the axioms of a net of local algebras of fields (in the vacuum representation).

Definition 2.70. A \mathbb{Z}_2 -graded AQFT is defined by a concrete C^* -algebra $\mathfrak{F} \subset \mathcal{B}(\mathcal{H})$, called the *algebra of fields*, and an assignment to every causally complete region $\mathcal{O} \in \mathcal{K}$ a C^* -subalgebra $\mathfrak{F}(\mathcal{O}) \subset \mathfrak{F}$, i.e.

$$\mathcal{O} \in \mathcal{K} \mapsto \mathfrak{F}(\mathcal{O}) \subset \mathfrak{F}. \quad (2.47)$$

which are called the *local algebras (of fields)*. This collection of local algebras must satisfy:

1. (*generating property*) $\mathfrak{F} = \overline{\bigcup_{\mathcal{O} \in \mathcal{K}} \mathfrak{F}(\mathcal{O})}^{\|\cdot\|}$.
2. (*isotony*) For any pair of regions $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{F}(\mathcal{O}_1) \subset \mathfrak{F}(\mathcal{O}_2)$.
3. (*causality*) There exists a \mathbb{Z}_2 -grading Γ such that for any pair of regions $\mathcal{O}_1 \times \mathcal{O}_2$, we have that $\mathfrak{F}(\mathcal{O}_1) \subset \mathfrak{F}(\mathcal{O}_2)'$.
4. (*Poincaré covariance*) There is a positive energy unitary representation $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow \mapsto U(\tilde{g}) \in \mathcal{B}(\mathcal{H})$ such that $U(\tilde{g}) \mathfrak{F}(\mathcal{O}) U(\tilde{g})^* = \mathfrak{F}(g\mathcal{O})$ for all $\mathcal{O} \in \mathcal{K}$ and $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow$, and such that it commutes with the grading, i.e. $[\Gamma, U(\tilde{g})] = 0$ for all $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow$.
5. (*vacuum*) There is a unique (up to a phase) unit vector $|0\rangle \in \mathcal{H}$, such that $U(\tilde{g})|0\rangle = |0\rangle$ for all $\tilde{g} \in \tilde{\mathcal{P}}_+^\uparrow$.
6. (*irreducibility*) $\mathfrak{F}' = \{\lambda \cdot \mathbf{1}\}$.

Any collection $\mathfrak{F}(\mathcal{O})$ of C^* -algebras satisfying the above axioms is called a *net of local algebras of fields*.

Remark 2.71. The above net is called algebra of fields or net of fields, in order to be distinguished from the algebra of observables or net of observables. Indeed, the term algebra of observables is reserved for a net where all its operators commute at spacelike distance, whereas the term algebra of fields emphasizes the fact that there are some operators, the “fermionic fields”, which anticommute at spacelike distance.

Remark 2.72. The above definition reduces to the axioms of a net of local algebras of observables in the vacuum representation (definitions 2.56 and 2.58) in the case when the grading is trivial, i.e. $\Gamma = \mathbf{1}$. In this sense, a net of fields is a generalization of a net of observables.

Remark 2.73. Under the assumption of weak additivity, Reeh-Schlieder theorem 2.70 also holds for the net of fields.

Given any net of fields $\mathfrak{F}(\mathcal{O})$, we can define its observable subnet just as $\mathfrak{A}(\mathcal{O}) := \{A \in \mathfrak{F}(\mathcal{O}) : \Gamma A \Gamma = A\}$. The observable subnet satisfies all the axioms of definitions 2.56 and 2.58. It seems to be that the observable subnet $\mathfrak{A}(\mathcal{O})$ has less information than the net of fields $\mathfrak{F}(\mathcal{O})$. However, one of the big success of the algebraic approach to QFT was to prove that all the physical relevant information is encoded in the observable algebra itself and that the field algebra $\mathfrak{F}(\mathcal{O})$ could indeed be reconstructed from $\mathfrak{A}(\mathcal{O})$. This is the famous Doplicher-Roberts reconstruction theorem [46, 50]. We will further discuss this topic in chapter 7.

2.2.5. Lattice structure of AQFT

According to proposition 2.51, the set of vN subfactors of $\mathcal{B}(\mathcal{H})$ form an orthocomplemented lattice. The following proposition shows that the class of causally complete regions also form an orthocomplemented lattice.

Proposition 2.74. *The set of causally complete regions \mathcal{K} has a natural structure of orthocomplemented lattice (ordered by inclusion) given by*

$$\emptyset \subset \mathcal{O} \subset \mathbb{R}^d, \quad (2.48)$$

$$\mathcal{O}_1 \vee \mathcal{O}_2 := (\mathcal{O}_1 \cup \mathcal{O}_2)'' , \quad (2.49)$$

$$\mathcal{O}_1 \wedge \mathcal{O}_2 := \mathcal{O}_1 \cap \mathcal{O}_2, \quad (2.50)$$

where $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ and \cup (resp. \cap) denote the union (resp. intersection) of sets. The complementation in the lattice is given by the causal complement $\mathcal{O} \mapsto \mathcal{O}'$.

We remark that the region $\mathcal{O}_1 \vee \mathcal{O}_2$ is in general much bigger than $\mathcal{O}_1 \cup \mathcal{O}_2$ (see figure 2.1). However, for strictly spacelike separated regions $\mathcal{O}_1 \bowtie \mathcal{O}_2$ we have $\mathcal{O}_1 \vee \mathcal{O}_2 = \mathcal{O}_1 \cup \mathcal{O}_2$. This is not longer true for non-strict spacelike separated regions. For example we have that $\mathcal{O} \vee \mathcal{O}' = \mathbb{R}^d$.

Thought it is not the most general scenario, we expect that a “complete” QFTs satisfied a stronger version of the axioms of AQFT.

Definition 2.75 (tentative). *A complete QFT in the vacuum sector \mathcal{H}_0 is given by a lattice homomorphism³⁶ \mathcal{A} from the lattice of causally complete regions \mathcal{K} into the lattice of vN subfactors of $\mathcal{B}(\mathcal{H}_0)$. In other words, is given by an assignment*

$$\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0) , \quad (2.51)$$

satisfying

³⁶See appendix A for the definition of lattice homomorphism.

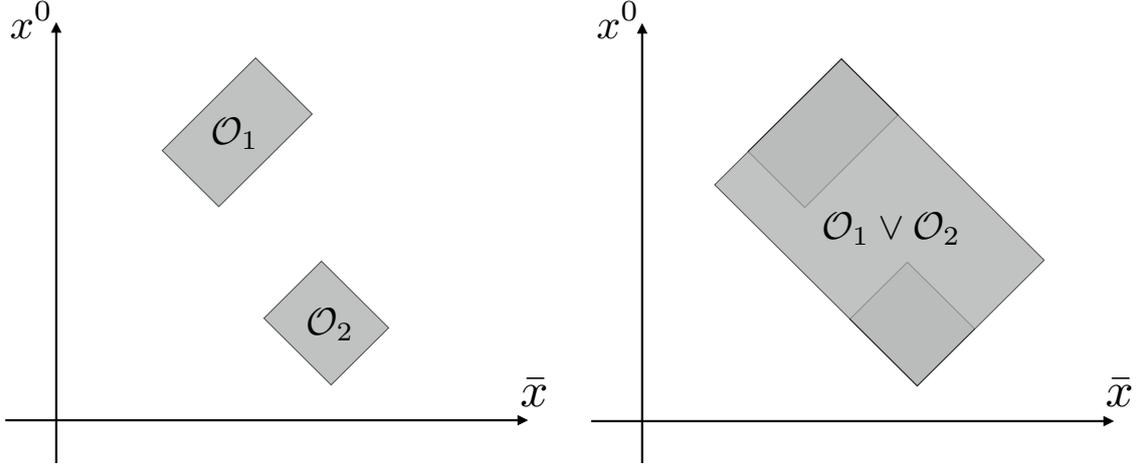


Figure 2.1: The supremum $\mathcal{O}_1 \vee \mathcal{O}_2$ of two causally complete regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$.

1. $\mathcal{A}(\emptyset) = \mathbf{1}$ and $\mathcal{A}(\mathbb{R}^d) = \mathcal{B}(\mathcal{H}_0)$,
2. $\mathcal{A}(\mathcal{O}_1 \vee \mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$,
3. $\mathcal{A}(\mathcal{O}_1 \wedge \mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1) \wedge \mathcal{A}(\mathcal{O}_2)$,
4. $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$,

for all $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$.

First of all, we are assuming that the local algebras of QFT are factors. At first glance, there is no fundamental reason to do that. However, under general circumstances it can be shown that the center $\mathcal{Z}(\mathcal{O})$ of a local algebra $\mathcal{A}(\mathcal{O})$ is a subalgebra of the center \mathcal{Z} of the global algebra \mathcal{A} [59]. Since the vacuum representation is irreducible (assumption 1 above), we have that \mathcal{Z} is trivial, and hence all local algebras must be factors.

The additivity property (assumption 2) is a stronger version than the one stated in section 2.2.1. This is because, as we explained above, the region $\mathcal{O}_1 \vee \mathcal{O}_2$ is in general much bigger than $\mathcal{O}_1 \cup \mathcal{O}_2$. Moreover, the additivity property stated in this way is too much strong that we do not expect it holds for many physically reasonable QFT models. For example, consider two small double cones \mathcal{O}_1 and \mathcal{O}_2 arranged along the time axis x^0 with vertices $\{(T - \varepsilon, \bar{0}), (T + \varepsilon, \bar{0})\}$ and $\{(-T - \varepsilon, \bar{0}), (-T + \varepsilon, \bar{0})\}$ respectively, with $T, \varepsilon > 0$ and $\frac{T}{\varepsilon} \gg 1$. Then, the region $\mathcal{O}_1 \vee \mathcal{O}_2$ is the double cone with vertices $\{(-T - \varepsilon, \bar{0}), (T + \varepsilon, \bar{0})\}$ (see figure 2.2). In this case, additivity means that the algebra generated by those two small double cones \mathcal{O}_1 and \mathcal{O}_2 coincides with the algebra of the big double cone $\mathcal{O}_1 \vee \mathcal{O}_2$. Such a behavior is not expected to be satisfied in all QFT models, but only for some sufficiently complete models. On the other hand, additivity is expected to hold in general for strictly spacelike separated regions, since in that case $\mathcal{O}_1 \vee \mathcal{O}_2 = \mathcal{O}_1 \cup \mathcal{O}_2$. Moreover, we expect that additivity also holds for a bigger class of regions, like the ones we exhibit in figure 2.3. These pairs of regions are characterized for being the Cauchy development of spacelike

regions lying in a common Cauchy surface. In lattice language, this is equivalent to the fact that \mathcal{O}_1 and \mathcal{O}_2 commute,³⁷ i.e.

$$\mathcal{O}_1 = (\mathcal{O}_1 \wedge \mathcal{O}_2) \vee (\mathcal{O}_1 \wedge \mathcal{O}'_2). \quad (2.52)$$

Furthermore, \mathcal{O}_1 and \mathcal{O}_2 commute if and only if $\mathcal{O}_1 \vee \mathcal{O}_2 = \text{int} (D(\overline{\mathcal{O}_1 \cup \mathcal{O}_2}))$.

According to the discussion above, assumption 2 above has to be weakened by

2'. $\mathcal{A}(\mathcal{O}_1 \vee \mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ whenever \mathcal{O}_1 and \mathcal{O}_2 commute.

In particular, additivity holds for complementary regions, i.e. $\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathbb{R}^d) = \mathcal{B}(\mathcal{H}_0)$. The price that we pay for replacing 2 by 2' is that the map (2.51) is no longer a lattice homomorphism. We also have to remark that not every QFT satisfies assumption 2'.

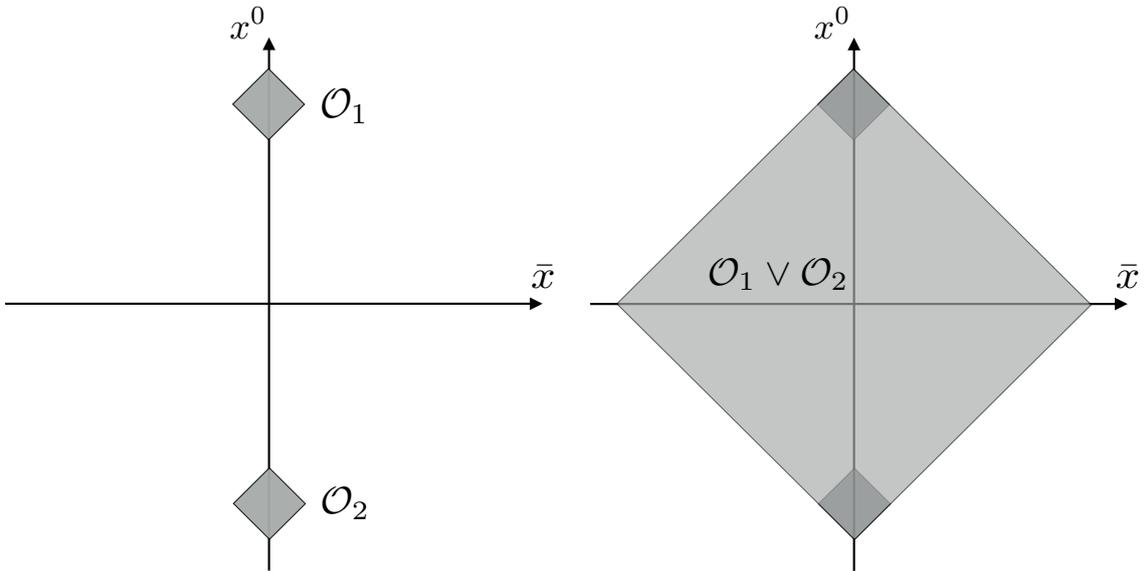


Figure 2.2: Additivity (assumption 2 in conjecture 2.75) is not usually expected to hold for this timelike separated regions.

Intersection property (assumption 3) has no counterpart in our previous definition 2.2.1. As additivity property (assumptions 2 or 2'), this property is not expected to hold for any QFT, but only for sufficiently complete models. Moreover, in an specific model, such properties may hold for some, but not all, pairs of causally complete regions. For example, in chapter 5 we exhibit a specific model where additivity holds for any pair of regions as the ones of figure 2.3. However, the same model does not satisfy additivity for strictly spacelike separated regions $\mathcal{O}_1 \times \mathcal{O}_2$.

Assumption 4 above is a stronger version of the causality axiom 3 in definition 2.56, which means $\mathcal{A}(\mathcal{O}') \subset \mathcal{A}(\mathcal{O})'$. In fact, assumption 4 is a kind of maximality condition for

³⁷See appendix A.

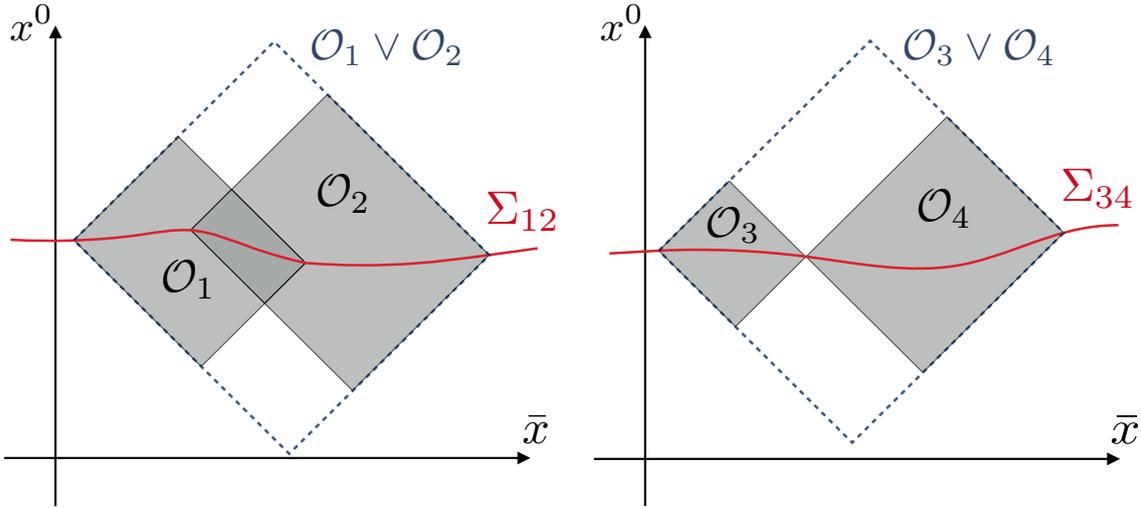


Figure 2.3: Additivity (assumption 2 in conjecture 2.75) it is expected to hold for \mathcal{O}_1 and \mathcal{O}_2 despite they are not strictly spacelike separated. This is because both these regions are the Cauchy development of spacelike regions lying in a common Cauchy surface Σ_{12} . The same holds for the regions \mathcal{O}_3 and \mathcal{O}_4 .

the local algebras, since $\mathcal{A}(\mathcal{O})'$ is the largest algebra that can be attached to region \mathcal{O} without spoiling causality. When assumption 4 holds for a region \mathcal{O} we say that the net algebra satisfies *duality* for region the \mathcal{O} . There are theories which satisfy duality for a large class of regions (e.g. the real scalar field, see chapter 5), whereas there other theories which satisfy duality just for regions that are topologically double cones (see chapters 6 and 7). One may be tempted to enlarge the local algebras defining $\mathcal{A}^d(\mathcal{O}) := \mathcal{A}(\mathcal{O})'$. The problem is when we replace simultaneously

$$\mathcal{A}(\mathcal{O}) \rightarrow \mathcal{A}^d(\mathcal{O}) = \mathcal{A}(\mathcal{O}')', \quad (2.53)$$

$$\mathcal{A}(\mathcal{O}') \rightarrow \mathcal{A}^d(\mathcal{O}') = \mathcal{A}(\mathcal{O})', \quad (2.54)$$

it could be the case that $[\mathcal{A}^d(\mathcal{O}), \mathcal{A}^d(\mathcal{O}')] \neq \{0\}$, because we have enlarged simultaneously both algebras. This motivates the following definition.

Definition 2.76. Let $\mathcal{A}(\mathcal{O})$ be a net of vN algebras. We say that $\mathcal{A}(\mathcal{O})$ satisfies *essential duality* if $\mathcal{A}^d(\mathcal{O})$ satisfies causality. If this happens, then $\mathcal{A}^d(\mathcal{O})$ satisfies duality for all $\mathcal{O} \in \mathcal{K}$.

When a net satisfies essential duality, i.e. the local algebras can be enlarged to obtain a net satisfying duality. However, it may happen that additivity and/or intersection property fail in the enlarged theory, even when in the original theory they are satisfied. In other words, there may be a balance between additivity and duality: non of them can be imposed without spoiling the other. A typical case is when the theory has non-trivial superselection sectors. We will further discuss this issue in chapter 7. On the other hand, essential duality may hold for some, but not all, regions. For example, in the case of spontaneous symmetry

breaking, it happens that the algebras attached to double cones do not satisfy duality but they do satisfy essential duality. Moreover, essential duality for double cones is a very fundamental property that might well be required of any ‘reasonable’ QFT [57].

Using the De Morgan laws,³⁸ we can formulate a different kinds of relations between additivity, intersection property and duality. For example, if unrestricted duality holds, then unrestricted additivity is equivalent to unrestricted intersection property. Also, we have that duality for a non-connected region with a finite number of components follows from duality for connected regions, unrestricted additivity and unrestricted intersection property.

Definition 2.75 just involves the algebraic relations between the local algebras of observables. We have to supplement them with a postulate concerning Poincaré covariance. Indeed, the lattice of causally complete regions and the lattice of vN subfactors of $\mathcal{B}(\mathcal{H}_0)$ are both \mathcal{P}_+^\uparrow -covariant lattices in the following sense.

Definition 2.77. Let L be an orthocomplemented lattice and G a group. We say that L is a G -covariant lattice if there exists a representation $\alpha : G \rightarrow \text{Aut}(L)$ of G into the lattice automorphism of L .³⁹

Proposition 2.78. Let \mathcal{K} be the lattice of causally complete regions. Then \mathcal{K} is a \mathcal{P}_+^\uparrow -covariant lattice when we consider the representation (2.34) of \mathcal{P}_+^\uparrow into $\text{Aut}(\mathcal{K})$.

Proposition 2.79. Let $g \mapsto U(g)$ be a unitary representation of \mathcal{P}_+^\uparrow on \mathcal{H}_0 . Then it defines a representation α of \mathcal{P}_+^\uparrow into the lattice of vN subalgebras of $\mathcal{B}(\mathcal{H}_0)$ by $\alpha_g(\mathcal{A}) := U(g)\mathcal{A}U(g)^*$. Then, the lattice of vN subalgebras of $\mathcal{B}(\mathcal{H}_0)$ is \mathcal{P}_+^\uparrow -covariant when we consider such a representation.

Definition 2.80. Let G be a group and L_1, L_2 be G -covariant lattices. A lattice homomorphism $\Phi : L_1 \rightarrow L_2$ is called G -covariant homomorphism if $\Phi \circ \alpha_g^{(L_1)} = \alpha_g^{(L_2)} \circ \Phi$ for all $g \in G$.

Finally, the assumptions 1 to 4 of conjecture 2.75, have to be complemented with the following assumption:

5. There is a positive energy unitary representation $g \in \mathcal{P}_+^\uparrow \mapsto U(g) \in \mathcal{B}(\mathcal{H}_0)$ such that the lattice homomorphism $\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0)$ is \mathcal{P}_+^\uparrow -covariant. There is also a unique (up to a phase) \mathcal{P}_+^\uparrow -invariant unit vector $|0\rangle \in \mathcal{H}_0$.

The lattice structure of conjecture 2.75 supplemented with assumption 5, gives an elegant correspondence between geometrical aspects (causally complete regions) and algebraic aspects (vN algebras) of QFT. However, this structure is too strong⁴⁰ that it would not hold for some concrete models with physical interest.

³⁸See appendix A.

³⁹See appendix A for the definition of lattice automorphism.

⁴⁰In particular, specially due to assumption 2.

On the other hand, the lattice structure could be (partially) restored if we work with causally complete regions within a common Cauchy surface. In this way, we fix a Cauchy surface $\Sigma \subset \mathbb{R}^d$ and we consider the set \mathcal{K}_Σ of all relative open sets of Σ . Then we have the following proposition.

Proposition 2.81. *Given any Cauchy surface $\Sigma \subset \mathbb{R}^d$, the set \mathcal{K}_Σ of relative open sets of Σ has a natural structure of orthocomplemented lattice (ordered by inclusion) given by*

$$\emptyset \subset \mathcal{C} \subset \Sigma, \quad (2.55)$$

$$\mathcal{C}_1 \vee \mathcal{C}_2 := \text{Int}(\overline{\mathcal{C}_1 \cup \mathcal{C}_2}), \quad (2.56)$$

$$\mathcal{C}_1 \wedge \mathcal{C}_2 := \mathcal{C}_1 \cap \mathcal{C}_2, \quad (2.57)$$

where $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2 \in \mathcal{K}$ and \cup (resp. \cap) denote the union (resp. intersection) of sets. The complementation in the lattice is given by the set complement $\mathcal{C}' := \Sigma - \overline{\mathcal{C}}$.

Definition 2.82. In the context of the above definition, the set \mathcal{C}' is usually called the *space complement* of \mathcal{C} .

Remark 2.83. The orthocomplemented lattice \mathcal{K}_Σ of the previous definition is also a *Boolean lattice*, i.e. every two elements of \mathcal{K}_Σ commute.⁴¹

Given $\mathcal{C} \in \mathcal{K}_\Sigma$, we have that $D(\mathcal{C}) \in \mathcal{K}$. Then we could define the net of vN factors

$$\mathcal{C} \in \mathcal{K}_\Sigma \mapsto \mathcal{A}_\Sigma(\mathcal{C}) := \mathcal{A}(D(\mathcal{C})) \subset \mathcal{B}(\mathcal{H}_0). \quad (2.58)$$

Assumptions 1, 2', 3 and 4 of definition 2.75 applied for the restricted set of double cones $\{D(\mathcal{C}) : \mathcal{C} \in \mathcal{K}_\Sigma\}$ gives place to a lattice homomorphism according to the following lemma.

Lemma 2.84. *Let $\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0)$ be net of vN factors satisfying the assumptions 1, 2', 3 and 4 of definition 2.75, and let $\Sigma \subset \mathbb{R}^d$ be a Cauchy surface. Then, the net*

$$\mathcal{C} \in \mathcal{K}_\Sigma \mapsto \mathcal{A}_\Sigma(\mathcal{C}) \subset \mathcal{B}(\mathcal{H}_0), \quad (2.59)$$

defined through 2.58 is a lattice homomorphism from the Boolean lattice of open regions \mathcal{K}_Σ into the lattice of vN subfactors of $\mathcal{B}(\mathcal{H}_0)$. In other words, the map (2.59) satisfies

1. $\mathcal{A}_\Sigma(\emptyset) = \mathbf{1}$ and $\mathcal{A}_\Sigma(\Sigma) = \mathcal{B}(\mathcal{H}_0)$,
2. $\mathcal{A}_\Sigma(\mathcal{C}_1 \vee \mathcal{C}_2) = \mathcal{A}_\Sigma(\mathcal{C}_1) \vee \mathcal{A}_\Sigma(\mathcal{C}_2)$,
3. $\mathcal{A}_\Sigma(\mathcal{C}_1 \wedge \mathcal{C}_2) = \mathcal{A}_\Sigma(\mathcal{C}_1) \wedge \mathcal{A}_\Sigma(\mathcal{C}_2)$,

⁴¹See appendix A for the definition of Boolean lattice automorphism.

$$4. \mathcal{A}_\Sigma(\mathcal{C}') = \mathcal{A}_\Sigma(\mathcal{C})' ,$$

for all $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2 \in \mathcal{K}_\Sigma$.

Since every Boolean sublattice $\mathcal{K}_S \subset \mathcal{K}$ is included in \mathcal{K}_Σ for some Cauchy surface Σ , we can say that (2.51) is a lattice homomorphism for every Boolean sublattice of \mathcal{K} .⁴²

To end, we again remark that the axioms of definition 2.75, or its weaker version summarized in lemma 2.84 may not hold for any QFT model. For example, the free massive scalar field satisfies properties 1, 2', and 3-5 of definition 2.75, but it does not satisfy 2 (see chapter 5). In [2] there is a heuristic argument on why the lattice structure of lemma 2.84 should hold for quantum electrodynamics. In any case, it is a very interesting job to find models satisfying all the assumptions of definition 2.75.

⁴²See appendix A for the definition of sublattice.

Chapter 3

Quantum information theory in operator algebras

In this chapter, we study the general aspects of quantum information theory in operator algebras. The main goal is to define and study many quantum information measures that we will use frequently in this thesis. A quantum information measure is defined to be any function over the space of states of a quantum system that describes the statistical properties of the system as a whole. Quantum information measures play a central role in the description and quantification of entanglement. As we discuss in the introduction, entanglement is a property of bipartite quantum systems, and it is manifested by non-classical correlations between its subsystems. It is a purely quantum phenomenon and can be viewed as an experimental resource, which can be exploited to perform quantum computations, teleportations, or other often counter-intuitive experiments. The organization of the chapter is as follows. First, we introduce the notion of a quantum channel, which has to be thought of as any “allowed” experimental operation over a quantum system which is restricted according to physical considerations. Subsequently, we study and introduce many quantum information measures and their properties. Such measures are usually monotonic under quantum channels which, roughly speaking, mean that the correlations along the system cannot be increased, on average, applying such a type of operations. For convenience, we separate the study of information measures into two sections: measures for finite and general quantum systems. We outline the central role of the relative entropy as a fundamental information quantity for general quantum systems and its connection with the modular theory of vN algebras. Reaching the end of the chapter, we study a particular kind of quantum channels known as conditional expectations, which have an important relation with relative entropy. To conclude, we introduce the concept of entanglement on a bipartite system. We study its structure and the ways to quantify it (entanglement measures).

In this chapter all the algebras used are C^* -algebras. We will explicitly emphasize when

a C^* -algebra is also a vN algebra.

3.1. Quantum channels

The notion of quantum channel is a very useful concept in the study of information and entanglement measures in quantum information theory. The idea is to describe all the physical transformations $\mathcal{F} : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ between two quantum system, i.e. all the mathematical transformations that are allowed from physical considerations.

Definition 3.1. A linear map $\mathcal{F} : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ between two C^* -algebras is called *positive* if $\mathcal{F}(A) \geq 0$ whenever $A \geq 0$, \mathcal{F} is called *completely positive (cp)* if $\mathcal{F} \otimes id_n : \mathfrak{A}_1 \otimes M_n(\mathbb{C}) \rightarrow \mathfrak{A}_2 \otimes M_n(\mathbb{C})$ is positive as a map, where $id_n : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is the identity map, i.e. $id_n(B) = B$ for all $n \in \mathbb{N}$ and all $B \in M_n(\mathbb{C})$.¹ A (completely) positive map is called *normalized* if $\mathcal{F}(\mathbf{1}) = \mathbf{1}$.

A normalized positive map gives place to a map $\mathcal{F}^* : \mathfrak{A}_1^* \rightarrow \mathfrak{A}_2^*$ that transforms a state ω on \mathfrak{A}_1 into a state $\mathcal{F}^*\omega$ on \mathfrak{A}_2 by de formula

$$(\mathcal{F}^*\omega)(A) = \omega(\mathcal{F}(A)) . \quad (3.1)$$

In other words, $\mathcal{F}^* : \mathfrak{S}(\mathfrak{A}_1) \rightarrow \mathfrak{S}(\mathfrak{A}_2)$. Conversely, any map from the states on \mathfrak{A}_1 to states on \mathfrak{A}_2 arises from a normalized positive linear map in this way. It is for this reason, that any normalized positive map is called a *state transformation*. Complete positivity is motivated by quantum information theory when it is allowed to couple our system with an ancilla, evolve the system plus the ancilla as a closed system, and finally trace out the ancilla. A cp normalized map is the mathematical characterization of a *quantum channel*.

In addition to a quantum channel, one could perform measurements and post-select a sub-ensemble according to the results. For a vN measurement, given by projections $P_k \in \mathfrak{A}_1$ with $\sum_k P_k = \mathbf{1}$, we note that the maps $\mathcal{F}_k : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ given by $A \mapsto P_k A P_k$ are cp but not normalized because $0 \leq \mathcal{F}_k(\mathbf{1}) = P_k \leq \mathbf{1}$. Performing the measurement on a state ω we obtain the new state

$$\omega_k := \frac{1}{p_k} \mathcal{F}_k^* \omega , \quad (3.2)$$

with probability $p_k := \omega(P_k)$ when $p_k \geq 0$. A combination of quantum channels and measurements is called a *quantum operation* [63]. It is described by a family $\mathcal{F}_k : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ of cp maps such that $\sum_k \mathcal{F}_k(\mathbf{1}) = \mathbf{1}$. The composition of quantum channels (resp. quantum operations) is also a quantum channel (resp. quantum operation). There are many examples of quantum channels [55], between them we want to remark the automorphisms and

¹ $M_n(\mathbb{C}) := \mathbb{C}^{d_n \times d_n}$ denotes the algebra of all square matrices of size $n \times n$, which is indeed a finite dimensional C^* -algebra.

conditional expectations (section 3.4).

In the case of vN algebras, we only consider normal maps as in the following sense.

Definition 3.2. A linear map $\mathcal{F} : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ is said to be *normal* if $\mathcal{F}^*\omega$ is a normal state on \mathfrak{A}_2 state for any normal state ω on \mathfrak{A}_1 .

3.2. Entropies for finite quantum systems

A *finite quantum system* is any system described by a finite dimensional algebra. Since for finite algebras, the concepts of C^* and vN algebra are equivalent, throughout this section we call them simply algebras. Any finite algebra \mathfrak{A} is isomorphic to a direct sum of full matrix algebras

$$\mathfrak{A} \simeq \bigoplus_{n=1}^N M_{d_n}(\mathbb{C}). \quad (3.3)$$

Such an algebra acts naturally in the Hilbert space $\mathcal{H} := \bigoplus_{n=1}^N \mathbb{C}^{d_n}$. We remark that the algebra \mathfrak{A} as an abstract object, could be represented in many different ways, which are all C^* -isomorphic but not unitarily equivalent. Any other faithful representation of the algebra \mathfrak{A} is of the form

$$\mathfrak{A} \simeq \bigoplus_{n=1}^N M_{d_n}(\mathbb{C}) \otimes \mathbf{1}_{f_n}, \quad (3.4)$$

where $\mathbf{1}_{f_n}$ is the (trivial) algebra of scalar matrices of size $f_n \times f_n$, and $f_n \in \mathbb{N}$ are arbitrary dimensions.² It is not difficult to see that (3.3) and (3.4) are isomorphic as algebras, but they are not (in general) unitarily equivalent because they act in finite dimensional vector spaces of different dimensions.

Among all the representations of the algebra \mathfrak{A} , the representation (3.3) is unique (up to unitary equivalence). It is called the *canonical representation*. The important fact of such a representation arises when one wants to compute some quantum information quantity, e.g. vN entropy, relative entropy, etc. Such quantities depend, in general, on a collection of algebras $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ and a collection of states $\omega_1, \dots, \omega_l$ on such algebras, and they are expected to depend only on them, and not on the matrix representation used to write down the expressions. However, as we see below, such information quantities are defined through explicit formulas that only hold when they are used in the canonical representation. In other words, when one uses a formula to compute any information quantity, one always has to use the canonical representation.

On the other hand, the representation matters when we consider an inclusion of algebras, i.e. when the algebra \mathfrak{A} is a subalgebra of a bigger algebra. For example, the algebra (3.3) is a subalgebra of $M_d(\mathbb{C})$ with $d := \sum_{n=1}^N d_n$, whereas the algebra (3.4) is a

²Expression (3.4) reduces (3.3) when $f_n = 1$ for all $n = 1, \dots, N$.

subalgebra of $M_{\tilde{d}}(\mathbb{C})$ with $\tilde{d} := \sum_{n=1}^N d_n f_n$. The set of all subalgebras of $M_{\tilde{d}}(\mathbb{C})$ is bigger than the set of all subalgebras $M_d(\mathbb{C})$. In particular, the commutants \mathfrak{A}' of (3.3) and (3.4) (considered them as vN algebras embedded in the bigger algebras $M_d(\mathbb{C})$ and $M_{\tilde{d}}(\mathbb{C})$) are respectively

$$\bigoplus_{n=1}^N \mathbf{1}_{d_n} \text{ and } \bigoplus_{n=1}^N \mathbf{1}_{d_n} \otimes M_{f_n}(\mathbb{C}), \quad (3.5)$$

which are not C^* -isomorphic, and hence their canonical representations are very different. This matters in the discussion about entanglement in quantum systems. There, not only the subsystem considered matters, i.e. \mathfrak{A} , but also the ‘‘complement’’ of our subsystem with respect we want to quantify the entanglement, i.e. \mathfrak{A}' .

For an algebra \mathfrak{A} in its canonical representation (3.3), we can define its *trace* as

$$\mathrm{Tr}_{\mathfrak{A}}(A_1 \oplus \cdots \oplus A_N) = \sum_{n=1}^N \mathrm{Tr}_{d_n}(A_n), \quad (3.6)$$

where Tr_{d_n} is the usual trace in $M_{d_n}(\mathbb{C})$. Any trace will be denoted simply as Tr when there is no confusion about the algebra involved. The *canonical dimension* of \mathfrak{A} is defined as $d_{\mathfrak{A}} := \sum_{n=1}^N d_n$, i.e. the dimension of the Hilbert space where the canonical representation acts. It is a general fact that for any state ω on \mathfrak{A} , there exists a unique density matrix $\rho_{\omega} \in \mathfrak{A}$ (in its canonical representation) such that

$$\omega(A) = \mathrm{Tr}_{\mathfrak{A}}(\rho_{\omega} A), \quad \forall A \in \mathfrak{A}, \quad (3.7)$$

which is called the *statistical operator* of ω . Conversely, any statistical operator $\rho \in \mathfrak{A}$ gives place to a unique state through the relation (3.7). In other words, the relation between states and statistical operators is one-to-one. The state corresponding to the statistical operator $\frac{1}{d_{\mathfrak{A}}}$ is called the *tracial state* or *maximally mixed state*, and it is denoted by $\tau_{\mathfrak{A}}$ or τ . Any statistical operator $\rho \in \mathfrak{A}$ can be uniquely decomposed as

$$\rho = \bigoplus_{n=1}^N p_n \rho_n, \quad (3.8)$$

where ρ_n are statistical operators on $M_{d_n}(\mathbb{C})$ and $p_1, \dots, p_N \geq 0$ such that $\sum_{n=1}^N p_n = 1$.

It is important to emphasize that the trace could be thought of as an abstract object, depending only on \mathfrak{A} , but to compute it, we must use the canonical representation. Whereas, a statistical operator is an object that only make sense in the canonical representation. For any other representation, many different density matrices are satisfying a relation like (3.7). As a matter of fact, given two subalgebras $\mathfrak{A}_1, \mathfrak{A}_2 \subset M_d(\mathbb{C})$ and a state ω on $M_d(\mathbb{C})$, we can consider the states $\omega_1 := \omega|_{\mathfrak{A}_1}$ and $\omega_2 := \omega|_{\mathfrak{A}_2}$, which are the

restrictions of ω to \mathfrak{A}_1 and \mathfrak{A}_2 . The statistical operators ρ_{ω_1} and ρ_{ω_2} of such states are (in general) matrices with different number of blocks and sizes.

Finally, a short comment about quantum channels in the finite dimensional case. Even in this case, the general structure of cp normalized maps is still poorly understood. However, in the simple case when $\mathfrak{A}_j := M_{d_j}(\mathbb{C})$ and $\mathcal{F} : M_{d_2}(\mathbb{C}) \rightarrow M_{d_1}(\mathbb{C})$, we have the usual representation in term of Krauss operators [64]

$$\mathcal{F}(A) = \sum_{j=1}^n V_j^\dagger A V_j, \text{ with } V_j \in \mathbb{C}^{d_2 \times d_1} \text{ and } \sum_{j=1}^n V_j^\dagger V_j = \mathbf{1}_{d_1}. \quad (3.9)$$

3.2.1. von Neumann entropy

Throughout this section, \mathfrak{A} is a finite dimensional algebra, $\omega \in \mathfrak{S}(\mathfrak{A})$ a state, and ρ_ω is its corresponding statistical operator (in the canonical representation).

Definition 3.3. The *von Neumann entropy* of the state ω on the algebra \mathfrak{A} is defined as

$$S_{\mathfrak{A}}(\omega) := -\text{Tr}_{\mathfrak{A}}(\rho_\omega \log \rho_\omega) = -\sum_{j=1}^n \lambda_j \log(\lambda_j), \quad (3.10)$$

where $\lambda_j \geq 0$ are the eigenvalues of ρ_ω . It will be denoted as $S(\omega)$ whenever there is no confusion about the algebra involved.

The vN entropy is the quantum counterpart of the Shannon entropy. The Shannon entropy of a classical probability distribution (p_1, \dots, p_n) is defined as

$$H(p_1, \dots, p_n) := -\sum_{j=1}^n p_j \log(p_j). \quad (3.11)$$

It was Shannon who first interpreted the above quantity as information. He showed that, up to a constant factor, (3.11) is the only function of the probabilities p_1, \dots, p_n that satisfies some natural postulates for an information quantity [65]. The quantity (3.11) may be thought of as uncertainty because we are unable to predict exactly the outcomes of an experiment having different outcomes with probabilities p_1, \dots, p_n . On the other hand, once we have performed the experiment, the amount of information we gained is expressed by (3.11). Hence, we shall regard uncertainty and information as complementary concepts. Shannon entropy appears in classical information theory as a measure of the theoretical optimal capacity to compress a classical message (alphabet) without losing information [66]. As a counterpart, vN entropy has the same meaning in quantum information theory [67]. However, the original interpretation of (3.10) was thermodynamical. Von Neumann used a gedanken experiment about the process of a separation of a gas formed by two different species. Computing the change on the entropy using the thermodynamical entropy,

he arrived at a formula for the entropy under the mixing of states. Then, assuming that the entropy for pure states vanishes, he finally obtained (3.10). Nowadays, it is known that expression (3.10) could be deduced from some postulates in an axiomatic way [65].

The following proposition summarizes the most relevant properties of the vN entropy.

Proposition 3.4. *The von Neumann entropy satisfies the following properties [65].*

1. (positivity) $S(\omega) \geq 0$ and $S(\omega) = 0$ if and only if ω is pure.
2. (boundedness) $S(\omega) \leq \log(d_{\mathfrak{A}})$ where $d_{\mathfrak{A}}$ is the canonical dimension of \mathfrak{A} , and $S(\omega) = \log(d_{\mathfrak{A}})$ if and only if $\omega = \tau_{\mathfrak{A}}$.
3. (continuity) The map $\omega \mapsto S(\omega)$ is continuous in the space of states.
4. (invariance) If α is an automorphism of \mathfrak{A} , then $S(\omega) = S(\omega \circ \alpha)$.
5. (mixing of disjoint states) If $\omega_1, \dots, \omega_n$ are pairwise disjoint states³ and $\sum_{j=1}^n p_j = 1$, then

$$S\left(\sum_{j=1}^n p_j \omega_j\right) = \sum_{j=1}^n p_j S(\omega_j) + H(p_1, \dots, p_n). \quad (3.12)$$

6. (concavity) If $\sum_{j=1}^n p_j = 1$, then

$$\sum_{j=1}^n p_j S(\omega_j) \leq S\left(\sum_{j=1}^n p_j \omega_j\right) \leq \sum_{j=1}^n p_j S(\omega_j) + H(p_1, \dots, p_n). \quad (3.13)$$

Now we state the properties of the vN entropy under tensor products of algebras. For that, it is useful to introduce the following definition, which will be used repeatedly along this thesis.

Definition 3.5. Given any two states ω_1, ω_2 on the algebras $\mathfrak{A}_1, \mathfrak{A}_2$, we define the *product state* $\omega_1 \otimes \omega_2$ in $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ as

$$\omega_1 \otimes \omega_2 (A_1 \otimes A_2) := \omega_1 (A_1) \omega_2 (A_2), \quad (3.14)$$

where $A_i \in \mathfrak{A}_i$ and (3.14) is extended by linearity to more general operators.

When $\mathfrak{A}_1, \mathfrak{A}_2$ are finite dimensional algebras, it is not difficult to see that if ρ_1, ρ_2 are the statistical operators of ω_1, ω_2 then $\rho_1 \otimes \rho_2$ is the statistical operator of $\omega_1 \otimes \omega_2$.⁴

³Two states ω_1, ω_2 are said to be disjoint states if their corresponding density ρ_1, ρ_2 matrices has disjoint supports. The support of a density matrix is the orthogonal complement of its kernel, or equivalently, is the span of its eigenvectors corresponding to non-zero eigenvalues.

⁴The canonical representation of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is (up to unitary equivalence) the tensor product of the canonical representations of \mathfrak{A}_1 and \mathfrak{A}_2 .

Proposition 3.6. *The von Neumann entropy satisfies the following properties [65].*

1. (additivity) If $\omega_i \in \mathfrak{S}(\mathfrak{A}_i)$, then $S(\omega_1 \otimes \omega_2) = S(\omega_1) + S(\omega_2)$.
2. (subadditivity) If $\omega_{12} \in \mathfrak{S}(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ and $\omega_i := \omega_{12}|_{\mathfrak{A}_i}$, then $S(\omega_{12}) \leq S(\omega_1) + S(\omega_2)$.
3. (strong subadditivity) If $\omega_{123} \in \mathfrak{S}(\mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \mathfrak{A}_3)$, $\omega_2 := \omega_{123}|_{\mathfrak{A}_2}$, $\omega_{12} := \omega_{123}|_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ and $\omega_3 := \omega_{123}|_{\mathfrak{A}_2 \otimes \mathfrak{A}_3}$, then

$$S(\omega_{123}) + S(\omega_2) \leq S(\omega_{12}) + S(\omega_{23}). \quad (3.15)$$

Now, we state a lemma about the equality of the vN entropy for complementary algebras.

Lemma 3.7. *Let $\mathfrak{A} \subset M_d(\mathbb{C})$ be a finite quantum subsystem of $M_d(\mathbb{C})$ and ω a pure state on $M_d(\mathbb{C})$.⁵ Then $S_{\mathfrak{A}}(\omega) = S_{\mathfrak{A}'}(\omega)$.*

Finally, let $\mathcal{F} : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ be a quantum channel and $\omega \in \mathfrak{S}(\mathfrak{A}_1)$. It is not always true that the vN entropy is monotonic under quantum channels in the following sense⁶

$$S_{\mathfrak{A}_1}(\omega) \leq S_{\mathfrak{A}_2}(\mathcal{F}^*\omega). \quad (3.16)$$

However, the relation (3.16) holds for a subset of quantum channels, called *double stochastic maps* [65]. They are characterized by the extra property: $\text{Tr}_{\mathfrak{A}_1}(\mathcal{F}(A)) = \text{Tr}_{\mathfrak{A}_2}(A)$ for all $A \in \mathfrak{A}_2$.⁷ In particular, any quantum channel $\mathcal{F} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is double stochastic.

3.2.2. Rényi entropy

A concept close to von Neumann entropy is the Rényi entropy.

Definition 3.8. *The Rényi entropy of order α ($\alpha > 0$ and $\alpha \neq 1$) of the state ω on the algebra \mathfrak{A} is defined as*

$$S_{\alpha, \mathfrak{A}}(\omega) := \frac{1}{1-\alpha} \log \text{Tr}_{\mathfrak{A}}(\rho_{\omega}^{\alpha}) = \frac{1}{1-\alpha} \log \left(\sum_{j=1}^n \lambda_j^{\alpha} \right), \quad (3.17)$$

⁵The set of pure states of a full matrix algebra $M_d(\mathbb{C}) = \mathbb{C}^{d \times d}$ is in one-to-one correspondence with the set of unit rays in the Hilbert space \mathbb{C}^d . See example 2.23.

⁶For example, consider $\mathfrak{A}_1 := M_d(\mathbb{C}) \oplus M_d(\mathbb{C})$, $\mathfrak{A}_2 := M_d(\mathbb{C})$ and a quantum channel $\mathcal{F} : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ given by $\mathcal{F}(A) = A \oplus A$. Any state ω on \mathfrak{A}_1 is represented by a statistical operator of the form $\rho_{\omega} = p_1 \rho_1 \oplus p_2 \rho_2$ where ρ_i are statistical operators in $M_d(\mathbb{C})$, $p_i \geq 0$ and $p_1 + p_2 = 1$. If we define ω_i as the states on $M_d(\mathbb{C})$ associated with the density matrices ρ_i , definition (3.10) gives $S_{\mathfrak{A}_1}(\omega) = p_1 S_{\mathfrak{A}_2}(\omega_1) + p_2 S_{\mathfrak{A}_2}(\omega_2) + H(p_1, p_2)$. On the other hand, it is not difficult to see that, the state $\mathcal{F}^*\omega \in \mathfrak{S}(\mathfrak{A}_2)$ is $\mathcal{F}^*\omega = p_1 \omega_1 + p_2 \omega_2$, which it has vN entropy $S_{\mathfrak{A}_2}(\mathcal{F}^*\omega) = S_{\mathfrak{A}_2}(p_1 \omega_1 + p_2 \omega_2)$. Then, the concavity of the vN entropy gives $S_{\mathfrak{A}_2}(\mathcal{F}^*\omega) \leq S_{\mathfrak{A}_1}(\omega)$. Moreover, it can always be chosen a state such as the strict inequality holds, which contradicts (3.16).

⁷The existence of a double stochastic map $\mathcal{F} : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$, implies that $d_{\mathfrak{A}_1} = d_{\mathfrak{A}_2}$.

where $\lambda_j \geq 0$ are the eigenvalues of ρ_ω . It will be denoted as $S_\alpha(\omega)$ whenever there is no confusion about the algebra involved.

The Rényi entropy satisfies the properties 1 to 4 of proposition 3.4, but it does not satisfy 5 and 6. It also satisfies additivity, but it does not satisfy subadditivity nor strong subadditivity. The Rényi entropy also satisfies the duality of lemma 3.7. In the limit when $\alpha \rightarrow 1$, the Rényi entropy becomes the vN entropy

$$\lim_{\alpha \rightarrow 1} S_\alpha(\omega) = S(\omega) . \quad (3.18)$$

This characteristic feature has been largely exploited in QFT since the replica trick gives an explicit expression for the Rényi entropy (for the vacuum state) in terms of a path integral.

3.2.3. Relative entropy

Throughout this section, \mathfrak{A} is a finite dimensional algebra, $\omega, \phi \in \mathfrak{S}(\mathfrak{A})$ two states, and ρ_ω, ρ_ϕ their corresponding statistical operators (in the canonical representation).

Definition 3.9. The *relative entropy* between the states ω, ϕ on the algebra \mathfrak{A} is defined as

$$S_{\mathfrak{A}}(\phi | \omega) := \text{Tr}_{\mathfrak{A}}(\rho_\phi(\log \rho_\phi - \log \rho_\omega)) . \quad (3.19)$$

It will be denoted as $S(\phi | \omega)$ whenever there is no confusion about the algebra involved.

The relative entropy is a measure of distinguishability between two states since it is always positive and it is zero only when both states are equal. The following proposition summarizes the most important properties of the relative entropy.

Proposition 3.10. *The relative entropy satisfies the following properties [65, 68].*

1. (*positivity*) $S(\phi | \omega) \geq 0$ and $S(\phi | \omega) = 0$ if and only if $\phi = \omega$.
2. (*unboundedness*) $S(\phi | \omega) = +\infty$ whenever $\text{supp}(\rho_\phi) \not\subset \text{supp}(\rho_\omega)$.
3. (*continuity*) The map $(\phi, \omega) \mapsto S(\phi | \omega)$ is continuous in the space of states, whenever it is finite.
4. (*invariance*) If α is an automorphism of \mathfrak{A} , then $S(\phi | \omega) = S(\phi \circ \alpha | \omega \circ \alpha)$.
5. (*convexity*) If $\sum_{n=1}^N p_n = 1$, then

$$\sum_{n=1}^N p_n S(\phi_n | \omega) \leq S\left(\sum_{n=1}^N p_n \phi_n | \omega_n\right) + H(p_1, \dots, p_n) . \quad (3.20)$$

6. (*joint-convexity*) If $\sum_{n=1}^N p_n = 1$, then

$$S \left(\sum_{n=1}^N p_n \phi_n \mid \sum_{n=1}^N p_n \omega_n \right) \leq \sum_{n=1}^N p_n S(\phi_n \mid \omega_n). \quad (3.21)$$

7. (*monotonicity*) If $\mathfrak{B} \subset \mathfrak{A}$ is a subalgebra, then $S_{\mathfrak{B}}(\phi \mid \omega) \leq S_{\mathfrak{A}}(\phi \mid \omega)$.

8. (*tensor products*) If $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$, $\omega \in \mathfrak{S}(\mathfrak{A})$, $\omega_j := \omega|_{\mathfrak{A}_j}$ and $\phi_j \in \mathfrak{S}(\mathfrak{A}_j)$ ($j = 1, 2$), then

$$S_{\mathfrak{A}}(\omega \mid \phi_1 \otimes \phi_2) = S_{\mathfrak{A}_1}(\omega_1 \mid \phi_1) + S_{\mathfrak{A}_2}(\omega_2 \mid \phi_2) + S_{\mathfrak{A}}(\omega \mid \omega_1 \otimes \omega_2). \quad (3.22)$$

9. (*quantum channels*) If $\mathcal{F} : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ is a quantum channel, then

$$S_{\mathfrak{A}_1}(\mathcal{F}^* \phi \mid \mathcal{F}^* \omega) \leq S_{\mathfrak{A}_2}(\phi \mid \omega). \quad (3.23)$$

As we claimed above, property 1 of proposition 3.10 says that relative entropy is a measure of distinguishability between two states. However, it is not a distance in the pure mathematical sense since it is not, in general, symmetric. In (3.19), it is useful to think ω as a known reference state against the unknown state ϕ is going to be compared. If ω is pure, the relative entropy is always infinite, unless $\phi = \omega$. This is because, in such a case, if $\rho_{\omega} = |\Omega\rangle\langle\Omega|$, then the projector $P := 1 - |\Omega\rangle\langle\Omega|$ has probability $\phi(P) \neq 0$. Therefore, if we measure such a projector in the state ϕ and we can repeat such an experiment as many times as we want, we can infer, with certainty, that $\phi \neq \omega$.

Relative entropy has also the following operational interpretation. Given the known reference state ω , the probability p of mistaken ϕ with ω after n judiciously chosen experimental measurements declines with n as $\sim e^{-nS(\phi|\omega)}$. In this sense, relative entropy is an experimentally accessible quantity (quantum Stein lemma [64, 69]).

Among all the properties of proposition 3.10, we want to remark 7 and 8. The monotonicity says that the two states are, in general, less indistinguishable if we restrict our system, i.e. \mathfrak{A} , to a subsystem, i.e. \mathfrak{B} . This is because in the subsystem there are less operators available, and hence, the number of experiments we can perform in order distinguish them decreases. In the same line, the quantum channel property asserts that we can never improve the capacity of distinguishing the two states performing physical operations. Moreover, any physical transformation of the states, in general, changes both states in a way that they become less distinguishable.

After a simple manipulation, expression (3.19) can be rewritten in the following way

$$S(\phi \mid \omega) = \Delta \langle K_{\omega} \rangle - \Delta S, \quad (3.24)$$

where

$$\Delta S = S(\phi) - S(\omega) , \quad \text{and} \quad \Delta \langle K_\omega \rangle = \phi(K_\omega) - \omega(K_\omega) . \quad (3.25)$$

The first equation in (3.25) is a difference of the vN entropies in the two states, whereas the second equation has to be interpreted as the difference of the expectation values of the operator $K_\omega := -\log(\rho_\omega)$. The operator K_ω is known as the “modular Hamiltonian” of the state ω , and it plays an important role in modular theory of vN algebras and the generalization of the relative entropy to general quantum systems (see sections (3.3.1) and (3.3.2) below). Relation (3.24) has been much used in QFT, because in many applications, it is easier to compute separately each term on the r.h.s. of (3.24) rather than the relative entropy (l.h.s.) itself.

3.2.4. Mutual information

Another quantity of great importance in quantum information theory is the mutual information. It is defined for a state ω on a bipartite system $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.

Definition 3.11. The *mutual information (MI)* of the state ω between \mathfrak{A}_1 and \mathfrak{A}_2 is defined as

$$I(\mathfrak{A}_1, \mathfrak{A}_2; \omega) := S_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}(\omega \mid \omega_1 \otimes \omega_2) , \quad (3.26)$$

where $\omega_i := \omega|_{\mathfrak{A}_i}$ are the restrictions of ω to \mathfrak{A}_i . It will be denoted by $I(\mathfrak{A}_1, \mathfrak{A}_2)$ or $I(\omega)$ whenever there is no confusion about the state or the algebras involved.

The mutual information is a measure of the information contained in the bipartite system $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, which is not contained in each subsystem \mathfrak{A}_i individually. If we interpret vN entropy as “information”, the interpretation of the MI as a “shared information” can be easily seen from the following relation

$$I(\mathfrak{A}_1, \mathfrak{A}_2; \omega) = S_{\mathfrak{A}_1}(\omega_1) + S_{\mathfrak{A}_2}(\omega_2) - S_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}(\omega) , \quad (3.27)$$

which holds for any finite quantum system. All the properties of the mutual information come from the ones of the relative entropy. The most salient are summarized in the following proposition.

Proposition 3.12. *The mutual information satisfies the following properties.*

1. (positivity) $I(\omega) \geq 0$ and $I(\omega) = 0$ if and only if $\omega = \omega_1 \otimes \omega_2$.
2. (boundedness) $I(\omega) \leq \log(d_{\mathfrak{A}_1}) + \log(d_{\mathfrak{A}_2})$ where $d_{\mathfrak{A}_i}$ is the canonical dimension of \mathfrak{A}_i , and $I(\omega) = \log(d_{\mathfrak{A}_1}) + \log(d_{\mathfrak{A}_2})$ if and only if ω is pure and $\omega_i = \tau_{\mathfrak{A}_i}$.
3. (continuity) The map $\omega \mapsto I(\omega)$ is continuous in the space of states.

4. (invariance) If α_i are automorphism of \mathfrak{A}_i and $\alpha := \alpha_1 \otimes \alpha_2$, then $I(\omega) = I(\omega \circ \alpha)$.
5. (monotonicity) if $\mathfrak{B}_i \subset \mathfrak{A}_i$ are subalgebras, then $I(\mathfrak{B}_1, \mathfrak{B}_2) \leq I(\mathfrak{A}_1, \mathfrak{A}_2)$.
6. (quantum channels) If $\mathcal{F}_j : \mathfrak{A}_{j,B} \rightarrow \mathfrak{A}_{j,A}$ ($j = 1, 2$) are quantum channels, and $\mathcal{F}_\otimes : \mathfrak{A}_{1,B} \otimes \mathfrak{A}_{2,B} \rightarrow \mathfrak{A}_{1,A} \otimes \mathfrak{A}_{2,A}$, then $I(\mathfrak{A}_{1,B}, \mathfrak{A}_{2,B}; \mathcal{F}_\otimes^* \omega) \leq I(\mathfrak{A}_{1,A}, \mathfrak{A}_{2,A}; \omega)$.

The mutual information can be considered as a measure of the correlations between both subsystems. In fact, the following inequality holds

$$I(\mathfrak{A}_1, \mathfrak{A}_2; \omega) \geq \frac{1}{2} \left(\frac{\omega(A_1 \otimes A_2) - \omega(A_1)\omega(A_2)}{\|A_1\| \|A_2\|} \right)^2, \quad (3.28)$$

for all $A_j \in \mathfrak{A}_j$ [15]. This means that the mutual information measures all the possible correlations between the two subsystems \mathfrak{A}_1 and \mathfrak{A}_2 for the state ω since it bounds all the connected correlators which measure such correlations, whatever they are quantum or classic.

3.3. Entropies for general quantum systems

In this section, we generalize all the formulas of the previous section for general (not necessarily finite) quantum systems. We use the term *general quantum system* to refer to any quantum system, which is described by a finite or an infinite dimensional algebra. The reason we do not reserve this section just for “infinite” quantum systems, is because all the formulas developed along this section also apply for finite quantum systems, and in that case, they coincide with the formulas stated in the previous section.

All the formulas of the previous section rely on the existence of a trace functional Tr on finite dimensional algebras. However, for a general infinite dimensional algebra, such a trace does not exist.⁸ There are some exceptions to the above statement. Among them, there are the type I factors, which are vN algebras isomorphic to $\mathcal{B}(\mathcal{H})$ where \mathcal{H} is some separable Hilbert space. For such algebras, the trace exists and is uniquely defined (up to a constant) as the usual trace in the Hilbert space. Then, given a normal state ω on $\mathcal{B}(\mathcal{H})$ represented by the density matrix ρ_ω , the expressions of the previous section apply. However, it can be shown, under general physical assumptions, that local algebras of QFT are not of this kind, but they are type III vN algebras [70]. Unfortunately, such algebras do not admit any trace functional. Therefore, we need to address this problem from a different perspective. This perspective is the modular theory of vN algebras.

We start this section reviewing the modular theory of vN algebras, from which the relative entropy is defined. Mutual information is then defined using relative entropy. At

⁸It is beyond the scope of this thesis to explain what a trace functional means and why it can not exist for general infinite dimensional algebras [2, 21].

the end of this section, we discuss why the vN entropy admits no definition for general quantum systems.

3.3.1. Modular theory

A vN algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is said to be in *standard form* if it has a standard vector $|\Psi\rangle \in \mathcal{H}$. It is an important fact, that any vN algebra is isomorphic to a unique (up to unitary equivalence) vN algebra in standard form. Such an algebra is called its *standard representation* [71]. Any normal state ω over a vN algebra in standard form has a *vector representative* $|\Omega\rangle \in \mathcal{H}$

$$\omega(A) = \langle \Omega | A | \Omega \rangle, \quad \forall A \in \mathcal{A}. \quad (3.29)$$

Thus, the set of normal states is already provided by the vector states and we do not have to resort to density matrices. Of course the correspondence $\omega \rightarrow |\Omega\rangle$ is one to many.⁹ Furthermore, when the normal state is also faithful, every vector representative is separating and there exists at least one representative which is also cyclic [2].

3.3.1.1. Modular Hamiltonian and modular flow

Throughout this section, $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a vN algebra and $|\Omega\rangle \in \mathcal{H}$ is a standard vector.

Fact 3.13. *There exists a unique (generally unbounded) closed antilinear operator S_Ω such that*

$$S_\Omega A |\Omega\rangle = A^* |\Omega\rangle, \quad \forall A \in \mathcal{A}. \quad (3.30)$$

Proof. See [21]. □

Definition 3.14. In the context of the above lemma, S_Ω is called the *modular involution* associated with the pair $\{\mathcal{A}, |\Omega\rangle\}$. Let $S_\Omega = J_\Omega \Delta_\Omega^{\frac{1}{2}}$ be its polar decomposition. Then, the (generally unbounded) positive self-adjoint Δ_Ω is called the *modular operator* and the antiunitary operator J_Ω is called the *modular conjugation*, the self-adjoint operator $K_\Omega := -\log(\Delta_\Omega)$ is called *modular Hamiltonian*, and the (strongly continuous) one-parameter group of unitaries Δ_Ω^{it} is called the *modular group*.

Remark 3.15. We remark that the modular Hamiltonian of the previous definition is not the same modular Hamiltonian we have introduced at the end of section 3.2.3. Roughly speaking, the later is like the “half” part of the former one. The modular Hamiltonian of the end of section 3.2.3 is only well-defined in the finite dimensional case, and it is an operator which belongs to the algebra considered. For such a reason, it is sometimes

⁹If $\mathcal{H}_\Omega = \overline{\mathcal{A}|\Omega\rangle}$ is the cyclic subspace of $|\Omega\rangle$ and $V \in \mathcal{A}'$ is a partial isometry with initial subspace \mathcal{H}_Ω , then the vector $V|\Omega\rangle$ is also a vector representative of ω . Furthermore, any other vector representative is of this form.

called the “inner” modular Hamiltonian. On the other hand, the modular Hamiltonian of definition 3.14 is always well-defined, but it does not belong to the algebra \mathcal{A} . Because of this, it is sometimes called the “full” modular Hamiltonian. The relation between these two objects will be clearer in example 3.18 below.

The following is the famous theorem of Tomita-Takesaki.

Theorem 3.16. (*Tomita-Takesaki*) *The modular conjugation J_Ω and the modular group Δ_Ω^{it} associated with $\{\mathcal{A}, |\Omega\rangle\}$ act as*

$$J_\Omega \mathcal{A} J_\Omega = \mathcal{A}', \quad (3.31)$$

$$\Delta_\Omega^{it} \mathcal{A} \Delta_\Omega^{-it} = \mathcal{A} \quad \text{and} \quad \Delta_\Omega^{it} \mathcal{A}' \Delta_\Omega^{-it} = \mathcal{A}', \quad (3.32)$$

for all $t \in \mathbb{R}$.

Proof. See [21]. □

The above lemma states that the modular group Δ_Ω^{it} induces a one-parameter group of automorphisms

$$\sigma_t^\Omega(A) := \Delta_\Omega^{it} A \Delta_\Omega^{-it}, \quad A \in \mathcal{A}, t \in \mathbb{R}, \quad (3.33)$$

on the algebra \mathcal{A} , which is called the *modular flow*.

The modular flow is a kind of dynamics or evolution in the algebra \mathcal{A} that is completely defined by a given state. It is usually called the *modular evolution*. However, it can be shown that the modular group Δ_Ω^{it} does not belong to \mathcal{A} neither \mathcal{A}' . Furthermore, it also can be shown that, in general, the modular flow is not *inner*, i.e. $\exists U(t) \in \mathcal{A}$ such that $\sigma_t^\Omega(A) = U(t) A U(t)^*$ for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. This means that the modular dynamics, which is uniquely determined by the state, involves operators which are outside the algebra.

Remark 3.17. The parameter t in 3.33 is usually called *modular time*. The theorem 3.16 means that any given state gives place to a natural notion of dynamics and hence a natural notion of time, both state-dependent. Without being rigorous, it suggests a kind of parallelism between modular theory and general relativity. In general relativity, the notion of time, which is encoded in the spacetime metric, is completely determined by the configuration of matter and energy along the spacetime, i.e. the state of the system. Moreover, modular theory is described in terms of operator algebras, i.e. quantum physics. This suggests that it could be a hidden connection between modular theory and quantum gravity. This is a proposal that is being explored nowadays.

Example 3.18. We show here how these definitions are realized in the finite dimensional case. Any finite dimensional vN factor in standard form is of the form $\mathcal{A} := M_n(\mathbb{C}) \otimes \mathbf{1}_n$. This algebra acts on the Hilbert space $\mathcal{H} := \mathbb{C}^n \otimes \mathbb{C}^n$ and it has a commutant $\mathcal{A}' = \mathbf{1}_n \otimes$

$M_n(\mathbb{C})$. A state ω on \mathcal{A} can be uniquely represented by a statistical operator $\rho_\omega \in M_n(\mathbb{C})$ such that $\omega(A \otimes \mathbf{1}) = \text{Tr}_n(\rho_\omega A)$. Let $\rho_\omega = \sum_{i=1}^n \lambda_i |i\rangle \langle i|$ be its spectral decomposition. The state ω is faithful if and only if $\lambda_i > 0$ for all $i = 1, \dots, n$. In such a case, the vector $|\Omega\rangle = \sum_{i=1}^n \sqrt{\lambda_i} |i\rangle \otimes |i\rangle$ is a standard vector representative of ω in \mathcal{A} . A straightforward computation shows that

$$S_\Omega(|i\rangle \otimes |j\rangle) = \sqrt{\frac{\lambda_i}{\lambda_j}} |i\rangle \otimes |j\rangle, \quad \text{and extended antilinearly to } \mathcal{H}, \quad (3.34)$$

$$\Delta_\Omega(|i\rangle \otimes |j\rangle) = \frac{\lambda_i}{\lambda_j} |i\rangle \otimes |j\rangle, \quad \text{and extended linearly to } \mathcal{H}, \quad (3.35)$$

$$J_\Omega(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle, \quad \text{and extended antilinearly to } \mathcal{H}. \quad (3.36)$$

Moreover, the modular operator (3.35) can be rewritten as $\Delta_\Omega = \rho_\omega \otimes \rho_\omega^{-1}$. Then we have that

$$\Delta_\Omega^{it} = \rho_\omega^{it} \otimes \rho_\omega^{-it}, \quad (3.37)$$

$$K_\Omega = -\log(\rho_\omega) \otimes \mathbf{1} + \mathbf{1} \otimes \log(\rho_\omega), \quad (3.38)$$

$$\sigma_t^\Omega(A \otimes \mathbf{1}) = \rho_\omega^{it} A \rho_\omega^{-it} \otimes \mathbf{1}. \quad (3.39)$$

Remark 3.19. The above example shows explicitly the relation between the “full” modular Hamiltonian $K_\Omega = -\log(\rho_\omega) \otimes \mathbf{1} + \mathbf{1} \otimes \log(\rho_\omega)$ and “inner” modular Hamiltonian $K_\omega = -\log(\rho_\omega)$. As we said in 3.15, the “inner” modular Hamiltonian has to be considered as half of the full modular Hamiltonian which is contained in the algebra.

Now we state the KMS-condition, which gives a sort of connection between thermal physics and modular evolution.

Definition 3.20. (KMS-condition) Let ω be a normal state and $\alpha_t \in \text{Aut}(\mathcal{A})$ a one-parameter group of automorphisms. We say that ω satisfies the *KMS-condition* at (inverse temperature) $\beta > 0$ with respect to α_t , if for any $A, B \in \mathcal{A}$, there exists a continuous function $G_{A,B} : \mathbb{R} + i[-\beta, 0] \rightarrow \mathbb{C}$, analytic on $\mathbb{R} + i(-\beta, 0)$ such that

$$G_{A,B}(t) = \omega(\alpha_t(A)B) \quad \text{and} \quad G_{A,B}(t - i\beta) = \omega(B\alpha_t(A)), \quad (3.40)$$

for all $t \in \mathbb{R}$.

Lemma 3.21. If ω satisfies the KMS-condition with respect to α_t , then $\omega(\alpha_t(A)) = \omega(A)$ for all $A \in \mathcal{A}$ and all $t \in \mathbb{R}$.

The KMS-condition is, within the algebraic setting, the way to characterize thermal states according to a specific dynamics. In other words, a thermal state (at inverse temperature β) for a given evolution (a one-parameter group of automorphisms) is a KMS-state.

The following example helps to understand better this concept.

Example 3.22. Let be the algebra $B(\mathcal{H})$ and a self-adjoint operator H acting \mathcal{H} . Let be $\beta > 0$ and suppose that the operator $e^{-\beta H}$ is a trace-class operator, i.e. $Z := \text{Tr}_{\mathcal{H}}(e^{-\beta H}) < \infty$. We may think of H as the Hamiltonian of the system which gives the usual evolution $\alpha_t(A) = e^{iHt} A e^{-iHt}$. The state $\omega_{\beta}(\cdot) := Z^{-1} \text{Tr}_{\mathcal{H}}(e^{-\beta H} \cdot)$ is a thermal state at inverse temperature β . Then, following definition 3.20, for any $A, B \in B(\mathcal{H})$ we define the function

$$G_{A,B}(t) := \omega(\alpha_t(A) B) = Z^{-1} \text{Tr}_{\mathcal{H}}(e^{-\beta H} e^{iHt} A e^{-iHt} B). \quad (3.41)$$

A straightforward computation shows

$$\begin{aligned} G_{A,B}(t - i\beta) &= Z^{-1} \text{Tr}_{\mathcal{H}}(e^{-\beta H} e^{iH(t-i\beta)} A e^{-iH(t-i\beta)} B) \\ &= Z^{-1} \text{Tr}_{\mathcal{H}}(e^{-\beta H} e^{iHt} e^{\beta H} A e^{-iHt} e^{-\beta H} B) \\ &= Z^{-1} \text{Tr}_{\mathcal{H}}(e^{-H\beta} B e^{iHt} A e^{-iHt}) \\ &= \omega(B\alpha_t(A)). \end{aligned} \quad (3.42)$$

Equation (3.42) shows that the state ω_{β} is a KMS-state (at inverse temperature β) for the evolution α_t .

The following theorem gives a connection between modular flow and KMS-condition.

Theorem 3.23. *The induced state $\omega(\cdot) = \langle \Omega | \cdot | \Omega \rangle$ satisfies the KMS-condition at $\beta = 1$ with respect to the modular flow σ_t^{Ω} .*

Proof. See [52]. □

The KMS-condition is quite restrictive as we see below.

Lemma 3.24. *Let σ_t^{Ω} be the modular flow associated with $\{\mathcal{A}, |\Omega\rangle\}$ and $\omega(\cdot) = \langle \Omega | \cdot | \Omega \rangle$ the induced state.*

- *If ω satisfies the KMS-condition at $\beta = 1$ with respect to some group of automorphisms α_t , then $\alpha_t = \sigma_t^{\Omega}$.*
- *Now suppose that \mathcal{A} is a factor. If a normal state ϕ satisfies the KMS-condition at $\beta = 1$ with respect to σ_t^{Ω} , then $\phi = \omega$.*

Proof. See [52]. □

3.3.1.2. Relative modular Hamiltonian and relative modular flow

The aim of this section is to define the relative modular operator, which plays a central role in the definition of the relative entropy of the next section. Throughout this section, $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a vN algebra and $|\Omega\rangle, |\Phi\rangle \in \mathcal{H}$ are two standard vectors.

Fact 3.25. *There exists a unique (generally unbounded) closed antilinear operator $S_{\Phi, \Omega}$ such that*

$$S_{\Phi, \Omega} A |\Omega\rangle = A^* |\Phi\rangle, \quad \forall A \in \mathcal{A}. \quad (3.43)$$

Proof. See [52]. □

Definition 3.26. In the context of the above lemma, $S_{\Phi, \Omega}$ is called the *relative modular involution* associated with the pair $\{\mathcal{A}, |\Omega\rangle, |\Phi\rangle\}$. Let $S_{\Phi, \Omega} = J_{\Phi, \Omega} \Delta_{\Phi, \Omega}^{\frac{1}{2}}$ be its polar decomposition. Then, the (generally unbounded) positive self-adjoint $\Delta_{\Phi, \Omega}$ is called the *relative modular operator* and the antiunitary operator $J_{\Phi, \Omega}$ is called the *relative modular conjugation*, the self-adjoint operator $K_{\Phi, \Omega} := -\log(\Delta_{\Phi, \Omega})$ is called *relative modular Hamiltonian*, and the (strongly continuous) one-parameter group of unitaries Δ_{Ω}^{it} is called *relative modular group*.

The following lemma shows a connection between the modular group and the relative modular group.

Lemma 3.27. *In the context of the above definition, we have that*

$$\Delta_{\Phi, \Omega}^{it} A \Delta_{\Phi, \Omega}^{-it} = \Delta_{\Phi}^{it} A \Delta_{\Phi}^{-it} \quad A \in \mathcal{A}, \quad (3.44)$$

$$\Delta_{\Phi, \Omega}^{it} A' \Delta_{\Phi, \Omega}^{-it} = \Delta_{\Omega}^{it} A' \Delta_{\Omega}^{-it} \quad A' \in \mathcal{A}', \quad (3.45)$$

for all $t \in \mathbb{R}$.

Proof. See [2]. □

As it happens with the modular group, $\Delta_{\Phi, \Omega}^{it} \notin \mathcal{A}, \mathcal{A}'$. However, due to the above lemma, the unitaries $u_{\Phi, \Omega}(t) := \Delta_{\Phi, \Omega}^{it} \Delta_{\Omega}^{-it}$ belong to \mathcal{A} for all $t \in \mathbb{R}$. such a family of unitaries $u_{\Phi, \Omega}(t)$ encodes most of the relevant information about the relative modular group. In fact, we have the following lemma.

Lemma 3.28. *The one-parameter family of unitaries¹⁰ $u_{\Phi, \Omega}(t) := \Delta_{\Phi, \Omega}^{it} \Delta_{\Omega}^{-it}$ satisfies the following conditions.*

1. $u_{\Phi, \Omega}(t) \in \mathcal{A}$ for all $t \in \mathbb{R}$.

¹⁰This one-parameter family of unitaries is not a one-parameter group.

2. $\sigma_t^\Phi(A) u_{\Phi,\Omega}(t) = u_{\Phi,\Omega}(t) \sigma_t^\Omega(A)$ for all $A \in \mathcal{A}$ and all $t \in \mathbb{R}$.
3. $u_{\Phi,\Omega}(t+t') = u_{\Phi,\Omega}(t) \sigma_t^\Omega(u_{\Phi,\Omega}(t'))$ for all $t, t' \in \mathbb{R}$.
4. There exists a continuous function $G : \mathbb{R} + i[-1, 0] \rightarrow \mathbb{C}$, analytic on $\mathbb{R} + i(-1, 0)$ such that $G(t) = \langle \Omega | u_{\Phi,\Omega}(t) | \Omega \rangle$ and $G(-i) = \langle \Phi | \Phi \rangle$.

Furthermore, when \mathcal{A} is a factor, $u_{\Phi,\Omega}(t)$ is uniquely determined by the conditions 1 to 4 above.

When the modular group Δ_Ω^{it} of the vector $|\Omega\rangle$ is known, the knowledge of $\Delta_{\Phi,\Omega}^{it}$ is equivalent to the knowledge of $u_{\Phi,\Omega}(t)$. However, in practical calculations, we may expect that the operators $u_{\Phi,\Omega}(t)$ are easier to be computed because of condition 1 in the previous lemma. We exploit this feature in chapter 5, when we use all these tools to compute the relative entropy for coherent states in free scalar theory.

Example 3.29. In the same context of example 3.18, now we have a second faithful state ϕ whose corresponding statistical operator is denoted by ρ_ϕ . As we have explained above, such a statistical operator has no non-zero eigenvalues. A straightforward computation shows that the relative modular operator has the expression $\Delta_{\Phi,\Omega}^{it} = \rho_\phi \otimes \rho_\omega^{-1}$, and hence we have that

$$\Delta_{\Phi,\Omega}^{it} = \rho_\phi^{it} \otimes \rho_\omega^{-it}, \quad (3.46)$$

$$K_{\Phi,\Omega} = -\log(\rho_\phi) \otimes \mathbf{1} + \mathbf{1} \otimes \log(\rho_\omega), \quad (3.47)$$

$$u_{\Phi,\Omega}(t) = \rho_\phi^{it} \rho_\omega^{-it} \otimes \mathbf{1}. \quad (3.48)$$

The condition 4 in lemma 3.28 is just a rewriting of the well-known relative KMS-condition. To state such a condition, we first introduce the following one-parameter family of automorphisms

$$\sigma_t^{\Phi,\Omega}(A) := \sigma_t^\Phi(A) u_{\Phi,\Omega}(t) = u_{\Phi,\Omega}(t) \sigma_t^\Omega(A), \quad A \in \mathcal{A}, t \in \mathbb{R}, \quad (3.49)$$

which are known as the *Connes Radon-Nikodym (CNR) cocycle*.

Theorem 3.30. (relative KMS-condition) *In the context of this section, let $\omega(\cdot) = \langle \Omega | \cdot | \Omega \rangle$ and $\phi(\cdot) = \langle \Phi | \cdot | \Phi \rangle$ be the induced states by $|\Omega\rangle$ and $|\Phi\rangle$. Then, for any given $A, B \in \mathcal{A}$, there exists a unique continuous function $G_{A,B} : \mathbb{R} + i[-1, 0] \rightarrow \mathbb{C}$, analytic on $\mathbb{R} + i(-1, 0)$ such that*

$$G_{A,B}(t) = \omega\left(\sigma_t^{\Phi,\Omega}(A) B\right) \quad \text{and} \quad G_{A,B}(t-i) = \phi\left(B \sigma_t^{\Phi,\Omega}(A)\right). \quad (3.50)$$

for all $t \in \mathbb{R}$.

3.3.2. Relative entropy for von Neumann algebras

The definition of the relative entropy for a general vN algebra is due to Araki [24].

Definition 3.31. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra in standard form, ω, ϕ two normal faithful states, and $|\Omega\rangle, |\Phi\rangle \in \mathcal{H}$ standard vector representatives. The *relative entropy (RE)* between the states ω, ϕ in the algebra \mathcal{A} is defined using the relative modular Hamiltonian $K_{\Omega, \Phi}$ as

$$S_{\mathcal{A}}(\phi | \omega) := \langle \Phi | K_{\Omega, \Phi} | \Phi \rangle . \quad (3.51)$$

It will be denoted as $S(\phi | \omega)$ whenever there is no confusion about the algebra involved.

Remark 3.32. When \mathcal{A} is not in standard form, the relative entropy is defined as above but computed over its standard representation [71]. When ω or ϕ are not faithful normal states, i.e. $|\Omega\rangle$ or $|\Phi\rangle$ are not standard, the definition is still possible but has to be somewhat modified [25, 65].

Remark 3.33. It can be shown that the r.h.s. of expression (3.51) is independent of the vector representatives chosen for the states [24]. This justifies the notation employed on the l.h.s. of (3.51).

Remark 3.34. When ω, ϕ are not normal states, the relative entropy could still be defined (see section 3.3.3).

The relative entropy (3.51) satisfies all the same properties that in the finite dimensional case, with the exception of 3 in proposition 3.10, which has to be replaced by the following one.

- 3'. (*lower-semicontinuity*) If $\lim_{n \rightarrow \infty} \|\phi_n - \phi\| = \lim_{n \rightarrow \infty} \|\omega_n - \omega\| = 0$, then $\liminf_{n \rightarrow \infty} S(\phi_n | \omega_n) \geq S(\phi | \omega)$.¹¹

The following lemma is useful for computations.

Lemma 3.35. *When the relative entropy is finite, in particular when $|\Omega\rangle$ belongs to the domain of $K_{\Omega, \Phi}$, the following expression holds*

$$S(\phi | \omega) = i \lim_{t \rightarrow 0} \frac{\langle \Phi | \Delta_{\Omega, \Phi}^{it} | \Phi \rangle - 1}{t} . \quad (3.52)$$

Proof. See [65]. □

Example 3.36. Let be ω, ϕ two faithful states on the finite quantum system $M_n(\mathbb{C})$. It is easy to see that the vN algebra $M_n(\mathbb{C})$ acting on \mathbb{C}^n is not in standard form. However, we can lift to its standard representation which becomes $\mathcal{A} = M_n(\mathbb{C}) \otimes \mathbf{1}_n$ as in the examples

¹¹ $\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (\inf_{m \geq n} x_m)$ denotes the *limit inferior*.

3.18 and 3.29. Then, all the formulas described in such examples apply here. In particular, if ρ_ω, ρ_ϕ are the statistical operators of ω, ϕ , we have that $K_{\Omega, \Phi} = -\log(\rho_\omega) \otimes \mathbf{1} + \mathbf{1} \otimes \log(\rho_\phi)$. Then, according to (3.51) we have that

$$\begin{aligned} S(\phi | \omega) &= \langle \Phi | K_{\Omega, \Phi} | \Phi \rangle = \langle \Phi | -\log(\rho_\omega) \otimes \mathbf{1} + \mathbf{1} \otimes \log(\rho_\phi) | \Phi \rangle \\ &= \langle \Phi | \mathbf{1} \otimes \log(\rho_\phi) | \Phi \rangle - \langle \Phi | \log(\rho_\omega) \otimes \mathbf{1} | \Phi \rangle . \end{aligned} \quad (3.53)$$

The second term is the expectation value of the operator $\log(\rho_\omega) \otimes \mathbf{1} \in M_n(\mathbb{C}) \otimes \mathbf{1}_n$ in the vector $|\Phi\rangle$. Since $|\Phi\rangle$ is a vector representative of ϕ , then we have that

$$\langle \Phi | \log(\rho_\omega) \otimes \mathbf{1} | \Phi \rangle = \phi(\log(\rho_\omega)) = \text{Tr}_n(\rho_\phi \log(\rho_\omega)) . \quad (3.54)$$

To compute the first term of (3.53), we have to use the expression of example 3.18, which shows how the vector representative $|\Phi\rangle$ is constructed in terms of the eigenvectors and eigenvalues of the statistical operator ρ_ϕ . Once we use such a relation, the computation is straightforward since $\mathbf{1} \otimes \log(\rho_\phi)$ and $|\Phi\rangle$ are diagonal on the same basis. In fact, we get

$$\langle \Phi | \mathbf{1} \otimes \log(\rho_\phi) | \Phi \rangle = \text{Tr}_n(\rho_\phi \log(\rho_\phi)) . \quad (3.55)$$

Replacing (3.54) and (3.55) into (3.53), we arrive to

$$S(\phi | \omega) = \text{Tr}_n(\rho_\phi(\log \rho_\phi - \log \rho_\omega)) , \quad (3.56)$$

which coincides with the definition of relative entropy for finite quantum systems (equation (3.19)).

3.3.3. Relative entropy for C^* -algebras

For general C^* -algebras, the relative entropy could be defined by three different but equivalent ways. Two of them use the definition 3.31 in an appropriate representation. The third one, Kosaki's formula, has no use of any representation and gives an expression involving only expectation values of the states in operators of the C^* -algebra. Of course, these three definitions are generalizations of definition 3.31, in the sense that they give the same result that (3.51) when they are applied to vN algebras and normal states.

Throughout this section, ω, ϕ are states on the C^* -algebra \mathfrak{A} . The relative entropy $S_{\mathfrak{A}}(\phi | \omega)$ is defined by any of the following three equivalent ways [65].

1. Using the definition 3.31 for the universal enveloping vN algebra $\tilde{\pi}(\mathfrak{A})''$, where $\tilde{\pi}$ is the universal representation of \mathfrak{A} .¹² In fact, any state ψ on \mathfrak{A} admits a unique normal

¹²The enveloping representation is defined as $\tilde{\pi} := \bigoplus_{\omega} \pi_{\omega}$ where the sum runs along all the states ω on

extension to $\tilde{\psi}$. Then we set

$$S_{\mathfrak{A}}(\phi \mid \omega) := S_{\tilde{\pi}(\mathfrak{A})''}(\tilde{\phi} \mid \tilde{\omega}). \quad (3.57)$$

2. Using the definition 3.31 for the algebra $\pi_{\omega}(\mathfrak{A})''$, where π_{ω} is the GNS-representation of ω . Of course, ω is a normal state for $\pi_{\omega}(\mathfrak{A})''$, but nothing guarantees that ϕ is normal state for $\pi_{\omega}(\mathfrak{A})''$. Then we set

$$S_{\mathfrak{A}}(\phi \mid \omega) := \begin{cases} S_{\pi_{\omega}(\mathfrak{A})''}(\phi \mid \omega) & \text{if } \phi \text{ is } \pi_{\omega}\text{-normal,} \\ +\infty & \text{otherwise.} \end{cases} \quad (3.58)$$

3. Kosaki's formula:

$$S_{\mathfrak{A}}(\phi \mid \omega) := \sup_{n \in \mathbb{N}} \sup_{x(t)} \left\{ \log(n) - \int_{\frac{1}{n}}^{+\infty} \frac{dt}{t} \left[\phi(x^*(t)x(t)) + \frac{1}{t} \omega(y^*(t)y(t)) \right] \right\}, \quad (3.59)$$

where the second sup is over all step functions $x : [\frac{1}{n}, +\infty) \rightarrow \mathfrak{A}$ such that $x(t) = \mathbf{0}$ for sufficiently large t and $y(t) := \mathbf{1} - x(t)$.

For computational purposes, expression 2 is easier to handle. Expression 3 has the advantage of being independent of any representation, and it is good to prove general statements concerning the relative entropy. Expression 1 was the original definition that was given by Araki [25]. As we claimed above, all these expressions give place to the same result, and it can be shown that they satisfy all the same properties which are satisfied by expression (3.51).

3.3.4. Mutual information

As it happens for finite quantum systems, the mutual information is defined for the case of a bipartite system $\mathcal{A}_1 \otimes \mathcal{A}_2$. It is generalized to general quantum systems though expression (3.26).

Definition 3.37. Let $\mathcal{A}_1, \mathcal{A}_2$ be two vN algebras and ω a normal faithful state in $\mathcal{A}_1 \otimes \mathcal{A}_2$. The *mutual information (MI)* of the state ω between \mathcal{A}_1 and \mathcal{A}_2 is defined as

$$I(\mathcal{A}_1, \mathcal{A}_2; \omega) := S_{\mathcal{A}_1 \otimes \mathcal{A}_2}(\omega \mid \omega_1 \otimes \omega_2), \quad (3.60)$$

where $\omega_i := \omega|_{\mathcal{A}_i}$ are the restrictions of ω to \mathcal{A}_i .¹³ It will be denoted by $I(\mathcal{A}_1, \mathcal{A}_2)$ or $I(\omega)$ whenever there is no confusion about the state or the algebras involved.

\mathfrak{A} and π_{ω} are their corresponding GNS-representation.

¹³The states ω_1, ω_2 and $\omega_1 \otimes \omega_2$ are normal whenever ω is normal.

The mutual information (3.60) satisfies all the same properties that in the finite dimensional case, with the exception of 3 in proposition 3.12, which has to be replaced by the following one.

3'. (*lower-semicontinuity*) If $\lim_{n \rightarrow \infty} \|\omega_n - \omega\| = 0$, then $\liminf_{n \rightarrow \infty} I(\omega_n) \geq I(\omega)$.

3.3.5. Comments about von Neumann entropy

The aim of this brief subsection is to briefly discuss why a generalization of the vN entropy does not exist for general quantum systems. The following are three partial arguments to support such a statement.

1. In the finite case, there are many mathematical relations between the relative entropy and the vN entropy. In fact, if one assumes that relative entropy as a more fundamental quantity, we are able to reconstruct the vN entropy (and all its properties) entirely from the relative entropy itself. On the other hand, we know that a rigorous definition of the relative entropy exists for general quantum systems and it satisfies all the same properties that the one in the finite case. Then, it is expected that if there should exist a generalization of the vN entropy, all the mathematical relations between RE and the vN entropy are preserved in the general case. However, in the finite case, the vN entropy can be expressed in term of the relative entropy as follows

$$S(\omega) := \sup \left\{ \sum_j p_j S(\omega_j | \omega) : \omega = \sum_j p_j \omega_j \text{ and } \sum_j p_j = 1 \right\}. \quad (3.61)$$

It can be shown, in many interesting examples, that such a supremum, in general, is $+\infty$.

2. The prototypical example of a general quantum system is QFT. In order to perform calculations, the continuum QFT may be regularized in a lattice introducing a UV cutoff ε . In such a way, the infinite dimensional algebras of QFT are converted into finite ones, and for the regularized theory, all the expressions of section 3.2 apply. Then, the vN entropy can be computed for the regularized theory obtaining a finite number $S_\varepsilon(\omega)$. However, in all interesting examples, we have that $S_\varepsilon(\omega) \rightarrow +\infty$ when $\varepsilon \rightarrow 0$.
3. Again in QFT, given the vacuum state ω and the algebra $\mathfrak{A}(\mathcal{O})$ attached to the spacetime region \mathcal{O} , we may want to compute the vN entropy $S_{\mathfrak{A}(\mathcal{O})}(\omega)$. Since the vacuum state is Poincaré invariant, we must have that $S_{\mathfrak{A}(\mathcal{O})}(\omega)$ is a Poincaré invariant quantity, i.e. $S_{\mathfrak{A}(g\mathcal{O})}(\omega) = S_{\mathfrak{A}(\mathcal{O})}(\omega)$ for all $g \in \mathcal{P}_+^\uparrow$. If we also require that $S_{\mathfrak{A}(\mathcal{O})}(\omega)$ satisfies strong subadditivity (which can be considered as the distinguished

“quantum” property of the vN entropy), then there is a theorem that asserts that $S_{\mathfrak{A}(\mathcal{O})}(\omega)$ cannot be non-negative for all \mathcal{O} , unless it is trivial (see section 4.2 below). This means that some of the properties of the vN must be relaxed in order to have a well defined and non-trivial vN entropy in QFT.

We will return to issues 2 and 3 in section 4.

3.4. Conditional expectations

Among all the normalized cp maps, there is a subclass called conditional expectations [64], which maps an algebra \mathfrak{A} onto a subalgebra $\mathfrak{B} \subset \mathfrak{A}$.

Definition 3.38. Let $\mathfrak{B} \subset \mathfrak{A}$ a subalgebra of \mathfrak{A} . A linear map $\varepsilon : \mathfrak{A} \rightarrow \mathfrak{B}$ is called a *conditional expectation* if it is positive, unit preserving and has the following property

$$\varepsilon(B_1 A B_2) = B_1 \varepsilon(A) B_2, \quad \forall A \in \mathfrak{A} \text{ and } \forall B_1, B_2 \in \mathfrak{B}. \quad (3.62)$$

Remark 3.39. Due to the positivity of ε , the relation $\varepsilon(BA) = B\varepsilon(A)$ for all $A \in \mathfrak{A}, B \in \mathfrak{B}$ implies $\varepsilon(AB) = \varepsilon(A)B$ for all $A \in \mathfrak{A}, B \in \mathfrak{B}$.

Remark 3.40. It can be shown that any conditional expectation is also completely positive.

Choosing $B_1 = A = \mathbf{1}$ in (3.62) we see that ε acts identically on \mathfrak{B} , and hence ε maps \mathfrak{A} onto the whole \mathfrak{B} . The physical meaning of a conditional expectation is as follows. Suppose we start with a physical system described by \mathfrak{A} , but for some reason (e.g. experimental limitations) we have at our disposal a subset of operators $\mathfrak{B} \subset \mathfrak{A}$. The conditional expectation $\varepsilon : \mathfrak{A} \rightarrow \mathfrak{B}$ is the mathematical realization of such a physical procedure, which is usually called a *coarse-graining*. In fact, as we argued in chapter 2, we should have no trouble in taking linear combinations and in composing operators in the subsystem, which means that $\mathfrak{B} \subset \mathfrak{A}$ is indeed a subalgebra. The conditional expectation also has to preserve elements in \mathfrak{B} because the expectation value of any operator $B \in \mathfrak{B}$ in any state $\omega \in \mathfrak{S}(\mathfrak{A})$ must be preserved by the coarse-graining, i.e. $\omega(\varepsilon(B)) = \omega(B)$. The positivity property is supported by interpretational considerations of quantum theory. The conditional expectation has to map observables (self-adjoint operators) and logical propositions (projectors) into observables and logical propositions. Moreover, the property (3.62) means that for the subsystem, the composition law of operators is preserved. It is needed because, as we have claimed, it should not be any experimental restriction to compose operators in \mathfrak{B} . The linearity is imposed to preserve the linear structure of states of spaces. Indeed, we expect that the expectation value $\omega(\varepsilon(\lambda_1 A_1 + \lambda_2 A_2))$ coincides with $\lambda_1 \omega(\varepsilon(A_1)) + \lambda_2 \omega(\varepsilon(A_2))$ for any state ω of \mathfrak{A} .

Given an algebra \mathfrak{A} and a subalgebra $\mathfrak{B} \subset \mathfrak{A}$, a conditional expectation from \mathfrak{A} to \mathfrak{B} may not exist and/or may not be unique [72]. The different possible conditional expectations correspond to different choices of how the operators in \mathfrak{A} are projected onto \mathfrak{B} . However, when \mathfrak{A} is a finite dimensional algebra (hence also \mathfrak{B}), there always exists a unique conditional expectation which satisfies

$$\mathrm{Tr}_{\mathfrak{A}}(\varepsilon(A)) = \mathrm{Tr}_{\mathfrak{A}}(A), \quad \forall A \in \mathfrak{A}. \quad (3.63)$$

Such a conditional expectation is called the *trace preserving conditional expectation* of the inclusion of algebras $\mathfrak{B} \subset \mathfrak{A}$.

Example 3.41. The prototypical example is when $\mathfrak{A} = M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ and $\mathfrak{B} = M_n(\mathbb{C}) \otimes \mathbf{1}_m$. The partial trace operation

$$\varepsilon(A \otimes B) = \mathrm{Tr}(B)(A \otimes \mathbf{1}), \quad (3.64)$$

is the (unique) trace preserving conditional expectation for such an inclusion of algebras.

A conditional expectation $\varepsilon : \mathfrak{A} \rightarrow \mathfrak{B}$ can be used to lift a state ϕ on \mathfrak{B} to the state $\phi \circ \varepsilon$ on \mathfrak{A} . There is also a close relation between conditional expectations and relative entropy. For any $\omega, \phi \in \mathfrak{S}(\mathfrak{A})$ such that $\phi = \phi \circ \varepsilon$, the following relation, called *conditional expectation property*, holds

$$S_{\mathfrak{A}}(\omega | \phi) - S_{\mathfrak{B}}(\omega|_{\mathfrak{B}} | \phi|_{\mathfrak{B}}) = S_{\mathfrak{A}}(\omega | \omega \circ \varepsilon). \quad (3.65)$$

The physical meaning of such a relation is as follows. The state $\phi = \phi \circ \varepsilon$ has to be considered a state on \mathfrak{A} whose physical content is indeed completely codified in \mathfrak{B} , or in other words, ϕ is a state which has no more correlations in \mathfrak{A} besides the correlations that are already present in \mathfrak{B} . Then, the difference of distinguishing any other state ω on \mathfrak{A} with such a state ϕ in the algebra \mathfrak{A} or in the algebra \mathfrak{B} (l.h.s. of (3.65)) is a quantity that depends only on the correlations of ω in \mathfrak{A} that are not present in \mathfrak{B} (r.h.s. of (3.65)). In particular, if we also have that $\omega = \omega \circ \varepsilon$, then

$$S_{\mathfrak{A}}(\omega | \phi) = S_{\mathfrak{B}}(\omega | \phi), \quad (3.66)$$

which means that both states are equally distinguishable in \mathfrak{A} and in \mathfrak{B} . Relation (3.65) plays a central role in our study of aspects of the entanglement entropy in QFTs that have superselection sectors (see chapter 7).

3.5. Entanglement

Along this chapter, we have studied many information measures both for finite and general quantum systems. Now, we will see how they are related to the concept of (quantum) entanglement. Some states in a bipartite quantum system show a particular kind of correlations, between the subsystems, which are possible only because of the quantum nature of the system. Quantum entanglement refers to the collection of such correlations, with no classical counterpart, as a whole, and we say that a state is entangled when at least “one” of these quantum correlations is present in such a state. In order to understand this phenomenon with more precision, we have to invoke Bell’s gedanken experiment [73, 74].

3.5.1. Bell’s inequality and entanglement entropy

We start with a physical bipartite system S formed by two subsystems S_A and S_B , each one attached to an observer, Alice and Bob. We may think that each subsystem is just a particle, hence we have two distinguishable particles. If the system is quantum mechanical, it is represented by a tensor product of C^* -algebras $\mathfrak{A} := \mathfrak{A}_A \otimes \mathfrak{A}_B$. Initially, the system S is prepared in a global state ω , and Alice and Bob perform measurements in each subsystem independently.¹⁴

We consider two different measurement devices on each subsystem given by observables $\{A_1, A_2\}$ on S_A and $\{B_1, B_2\}$ on S_B . For convenience, we label the outcomes of each measurement with the values $A_j, B_j \in \{-1, +1\}$. We are able to perform simultaneously any pair of measurements (A_j, B_k) , but we are not able to measure (A_1, A_2) nor (B_1, B_2) simultaneously. We also assume that we can reproduce the state ω with perfect accuracy as many times as we want and that the measurements can be performed with perfect accuracy.

We run the experiment $4N$ times, with $N \gg 1$. The first N times we measure (A_1, B_1) , the second N times we measure the (A_1, B_2) , and so on. We can estimate experimentally the expectation values $\langle A_j B_k \rangle$, and therefore we can estimate the expectation value of the observable

$$C := A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2. \quad (3.67)$$

Assume now that the bipartite system is classical. This means that the observables (A_1, A_2) and (B_1, B_2) are simultaneously measurable. According to the basic formulas of

¹⁴“Independently” means that if Alice performs any measurement in her subsystem, this will not modify the probabilities of the outcomes performed by Bob in his subsystem. In the classical case, this is always true. In the quantum case, it is only true (for a general state) whenever the operators, representing such measurements, commute. In our case, this holds because the subsystems are in a tensor product. Physically, it could be guaranteed whenever Alice and Bob are spacelike separated.

classical probability theory, a straightforward computation gives

$$|\langle C \rangle_\omega| \leq 2, \quad (3.68)$$

independent of the choice of the state ω and the observables, as long as they take values in the set $\{-1, +1\}$. Formula (3.68) is known as the *CHSH version of Bell's inequality*.¹⁵

However, if the bipartite system is quantum mechanical the above inequality can be violated. The prototypical example is the bipartite quantum system of two qubits. Each subsystem is represented by the finite dimensional algebras $\mathfrak{A}_A := \mathfrak{A}_B := M_2(\mathbb{C})$, and hence, the bipartite system is represented by $\mathfrak{A} := M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. Each subsystem acts naturally on the Hilbert space $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$, and we fix an orthonormal basis $\{|0\rangle, |1\rangle\}$ of such a Hilbert space. In order to show explicitly the violation of (3.68), we can choose observables of the form

$$A_j := \bar{\sigma} \cdot \bar{a}_j \otimes \mathbf{1}_2 \quad \text{and} \quad B_j := \mathbf{1}_2 \otimes \bar{\sigma} \cdot \bar{b}_j, \quad (3.69)$$

where $\bar{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ is the usual vector formed by the Pauli matrices, and $\bar{a}_j, \bar{b}_j \in \mathbb{R}^3$ with $|\bar{a}_j| = |\bar{b}_j| = 1$ so that the eigenvalues of such operators belong to $\{-1, +1\}$ as we need. In particular, taking

$$\bar{a}_1 := (0, 0, 1), \quad \bar{a}_2 := (1, 0, 0), \quad \bar{b}_1 := \frac{1}{\sqrt{2}}(1, 0, 1), \quad \bar{b}_2 := \frac{1}{\sqrt{2}}(1, 0, -1), \quad (3.70)$$

and the pure state ψ given by the vector (Bell pair)

$$|\Psi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \in \mathcal{H}_A \otimes \mathcal{H}_B, \quad (3.71)$$

a straightforward computation shows

$$\langle C \rangle_\psi := \langle \Psi | A_1 \otimes B_1 | \Psi \rangle + \langle \Psi | A_1 \otimes B_2 | \Psi \rangle + \langle \Psi | A_2 \otimes B_1 | \Psi \rangle - \langle \Psi | A_2 \otimes B_2 | \Psi \rangle = 2\sqrt{2} > 2. \quad (3.72)$$

The conclusion is that the state ψ exhibits correlations between the two subsystems that cannot be reproduced by a classical system. This is a pure quantum phenomenon, and it is possible because of the particular structure of the underlying state. A further computation shows that

$$|\langle C \rangle_\omega| \leq 2\sqrt{2}, \quad (3.73)$$

for any state ω and any observables having eigenvalues in the set $\{-1, +1\}$. This means that the state ψ gives a maximum violation of the Bell's inequality. This fact is usually

¹⁵CHSH stands for John Clauser, Michael Horne, Abner Shimony, and Richard Holt, who described it in the paper [74].

interpreted stating that ψ is a *maximally entangled state*.

It is important to emphasize that such correlations occur even if \mathfrak{A}_A and \mathfrak{A}_B are commuting algebras, i.e. they are statistically independent. In fact, the above behavior is only possible because of the existence of non-commuting observables within \mathfrak{A}_A and \mathfrak{A}_B . If instead of the above operators, we chose commuting operators $[A_1, A_2] = 0$ and $[B_1, B_2] = 0$, we would obtain an expectation value $|\langle C \rangle_\psi| \leq 2$, as in the classical case. This means that the quantum character of the system is fundamental to have entanglement. In a more general scenario, the entanglement depends not only on the state but also on the algebras \mathfrak{A}_A and \mathfrak{A}_B . As long as \mathfrak{A}_A and \mathfrak{A}_B are bigger, more operators are available within each subsystem, and hence more quantum correlations could be exhibited by the state. On the other hand, when the algebras \mathfrak{A}_A and \mathfrak{A}_B are Abelian, the inequality (3.68) is satisfied for any state, and hence there is no entanglement between the systems.

As we have already claimed, the form (3.71) of the state ψ is crucial to violate Bell's inequality. For example, the state ω , given by the statistical operator

$$\rho_\omega := p |00\rangle \langle 00| + (1 - p) |11\rangle \langle 11|, \quad p \in [0, 1], \quad (3.74)$$

satisfies $|\langle C \rangle_\omega| \leq 2$, for any choice of the operators A_j, B_k with eigenvalues $\{-1, +1\}$. The state ω may exhibit correlations but of the classical type.

The violation of Bell's inequality in a real experiment implies that the underlying system is governed by the laws of quantum mechanics. Many experimental tests were performed obtaining successful violations of Bell's inequality [75–89]. This pure quantum phenomenon has already caused a lot of problems to philosophers of quantum theory [90]. However, there is an almost general consensus that such a phenomenon does not violate Einstein's principle of locality. The non-local correlations, due to the entanglement, cannot be used to transmit information at spacelike distances. Moreover, from a conceptual viewpoint, the violation of Bell's inequality rules out the possibility of a quantum theory of local hidden variables compatible with Poincaré invariance. However, it leaves the door open to non-local hidden variables theories, such as De Broglie–Bohm theory, Many Worlds Theory, etc.

Any operation (unitary evolution and/or projective measurement) performed by Alice or Bob in each of their subsystems is called a local operation. If we also allow them to communicate classically (e.g. by a call phone), the operations allowed in the bipartite system are called *local operations with classical communication (LOCC)*. LOCC's have been used experimentally to teleport a quantum state along spacelike distances [91]. The entanglement has been largely exploited in quantum computation in order to show (until now in a theoretical way) that a quantum computer can solve many computational problems in a faster way than any classical computer does [67].

The question which naturally arises is if we can better understand the structure of the entanglement and if we can quantify it. The first attempt to do that was using the concept of *entanglement entropy* (*EE*). In the above context, it is given by the vN entropy of the reduced states

$$S_{E,\mathfrak{A}}(\omega) := S_{\mathfrak{A}_A}(\omega|_{\mathfrak{A}_A}) = S_{\mathfrak{A}_B}(\omega|_{\mathfrak{A}_B}), \quad (3.75)$$

where the second equality holds because the state ω is a pure state on \mathfrak{A} (lemma 3.7). For the state (3.71), we have that $S_{E,\mathfrak{A}}(\psi) = \log(2)$, which is the maximum value possible for this quantity (3.75). This agrees with the fact that ψ is a maximally entangled state since it gives maximum violation of Bell's inequality. The entanglement entropy is a good entanglement measure for pure states in finite quantum systems. However, it has (at least) two important problems: 1) it does not give a good measure of entanglement for non-pure global states, and 2) it is only well-defined for finite dimensional or type I vN algebras. The second problem arises naturally in QFT, as we will see in the following chapter. These facts force us to take alternative ways to define EE in QFT.

In the following subsections, we explain in more detail how the example above can be generalized to general quantum systems. We define with precision the concepts of entangled states, separable state, and entanglement measure.

3.5.2. Entangled and separable states

A general *bipartite* (quantum) system consists of two algebras $\mathfrak{A}_1, \mathfrak{A}_2$, called the *subsystems*, which give place to the algebra $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$, called the *system*. Any state $\omega \in \mathfrak{S}(\mathfrak{A})$ is called a *global state*, and for example, the term *global pure state* is frequently used to denote any element in $\mathfrak{S}_p(\mathfrak{A})$. Generally, the subsystems and the system are considered to be vN algebras, and all the states to be normal. We can now introduce the distinction between “separable states” and “entangled states”.

Definition 3.42. A global state $\omega \in \mathfrak{S}(\mathfrak{A})$ is said to be *separable* if it can be written as

$$\omega = \sum_{j=1}^{\infty} p_j \omega_{1,j} \otimes \omega_{2,j}, \quad (3.76)$$

for some collection of states $\omega_{i,j} \in \mathfrak{S}(\mathfrak{A}_i)$ and non-negative numbers $p_j \geq 0$ such that $\sum_{j=1}^{\infty} p_j = 1$ ($i = 1, 2$ and $j \in \mathbb{N}$).¹⁶ In particular, expression (3.76) means that $\omega(A_1 \otimes A_2) = \sum_{j=1}^{\infty} p_j \omega_{1,j}(A_1) \omega_{2,j}(A_2)$ for all $A_i \in \mathfrak{A}_i$. Any non-separable state is called *entangled*.

Since any mixed state is a convex combination of pure states, then any separable state is a convex combination of pure states. And of course, the converse is trivially true. Then, we do not lose any generality just considering separable states as in (3.76) where all $\omega_{i,j}$

¹⁶The sum (3.76) is considered as a uniform convergent series.

are pure. Depending on the global state ω , the outcomes of separate measurements on the two subsystems $\mathfrak{A}_1, \mathfrak{A}_2$ can exhibit different kinds of correlations. When $\omega := \omega_1 \otimes \omega_2$, there are no correlations at all. For a general separable state, however, there can be correlations, which are of a classical nature. Entangled states exhibit even more general “quantum” correlations.

3.5.3. Entanglement measures

The core of the entanglement measures lies in the notion of separable operation [55]. In section 3.1, we have introduced the set of quantum operations. Any operation is an arbitrary combination of quantum channels (cp normalized maps) and projective measurements. For any two given bipartite systems $\mathfrak{A}_{1,A} \otimes \mathfrak{A}_{2,A}$ and $\mathfrak{A}_{1,B} \otimes \mathfrak{A}_{2,B}$, we say that a cp map $\mathcal{F} : \mathfrak{A}_{1,B} \otimes \mathfrak{A}_{2,B} \rightarrow \mathfrak{A}_{1,A} \otimes \mathfrak{A}_{2,A}$ is *local* if there exist cp maps $\mathcal{F}_i : \mathfrak{A}_{i,B} \rightarrow \mathfrak{A}_{i,A}$ such that

$$\mathcal{F}(A_1 \otimes A_2) = \mathcal{F}_1(A_1) \otimes \mathcal{F}_2(A_2) = (\mathcal{F}_1 \otimes \mathcal{F}_2)(A_1 \otimes A_2) , \quad (3.77)$$

for all $A_i \in \mathfrak{A}_i$. More generally, a *separable operation* is by definition any map $\mathcal{F} : \mathfrak{A}_{1,B} \otimes \mathfrak{A}_{2,B} \rightarrow \mathfrak{A}_{1,A} \otimes \mathfrak{A}_{2,A}$ of the form $\mathcal{F} = \sum_{j=1}^n \mathcal{F}_j$ where each \mathcal{F}_j is a local cp map of the form (3.77) and $\mathcal{F}(\mathbf{1}) = \sum_{j=1}^n \mathcal{F}_j(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1}$. Such an operation maps a state ω of \mathfrak{A} , with probability $p_j := \omega(\mathcal{F}_j(\mathbf{1} \otimes \mathbf{1}))$, to the state $\frac{1}{p_j} \mathcal{F}_j^* \omega$.

The main idea of an entanglement measure is to give a partial order in the set of states of a bipartite system, which indicates which states are “more entangled” than other states. Until now, given a general bipartite system, there is no canonical or preferred way to do that. It could be because we have not already understood the structure of the entanglement in a final way, or perhaps, the different ways could be attached to different experimental choices. The important fact that they all have in common is that entanglement cannot be created performing local operations. The following definition is not definitive, but it collects all the properties we expect for an entanglement measure.

Definition 3.43. An entanglement measure E is a function from the set of all bipartite quantum systems $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ and their corresponding states $\mathfrak{S}(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ to the real numbers, i.e.

$$(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \omega) \mapsto E_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}(\omega) = E(\omega) , \quad (3.78)$$

which satisfies the following properties.

1. (*symmetry*) E is independent of the order of the systems \mathfrak{A}_1 and \mathfrak{A}_2 .
2. (*positivity*) $E(\omega) \geq 0$ and $E(\omega) = 0$ if and only if ω is separable.

3. (*convexity*) If $\omega = \sum_{j=1}^n p_j \omega_j$ with $p_j \geq 0$ and $\sum_{j=1}^n p_j = 1$, then

$$E(\omega) \leq \sum_{j=1}^n p_j E(\omega_j) . \quad (3.79)$$

4. (*monotonicity under separable operations*) For any separable operation $\mathcal{F} : \mathfrak{A}_{1,B} \otimes \mathfrak{A}_{2,B} \rightarrow \mathfrak{A}_{1,A} \otimes \mathfrak{A}_{2,A}$ of the form $\mathcal{F} = \sum_{j=1}^n \mathcal{F}_j$, then

$$\sum_{j=1}^n p_j E\left(\frac{\mathcal{F}_j^* \omega}{p_j}\right) \leq E(\omega) , \quad (3.80)$$

where the sum runs over all $p_j := \omega(\mathcal{F}_j(\mathbf{1})) > 0$.

5. (*superadditivity*) Let $\mathfrak{A}_1 = \mathfrak{A}_{1,A} \otimes \mathfrak{A}_{1,B}$ and $\mathfrak{A}_2 = \mathfrak{A}_{2,A} \otimes \mathfrak{A}_{2,B}$ be two bipartite systems, and let $\omega \in \mathfrak{S}(\mathfrak{A}_1 \otimes \mathfrak{A}_2)$ and let $\omega_A := \omega|_{\mathfrak{A}_{1,A} \otimes \mathfrak{A}_{2,A}}$ and $\omega_B := \omega|_{\mathfrak{A}_{1,B} \otimes \mathfrak{A}_{2,B}}$. Then

$$E_{\mathfrak{A}_{1,A} \otimes \mathfrak{A}_{2,A}}(\omega_A) + E_{\mathfrak{A}_{1,B} \otimes \mathfrak{A}_{2,B}}(\omega_B) \leq E_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}(\omega) , \quad (3.81)$$

and equality holds when $\omega = \omega_A \otimes \omega_B$.

Property 3 states that entanglement cannot be increased by mixing states, however, it can be reduced. Property 4 states that no entanglement can be won by performing separable operations. In particular if two states ω, ϕ are such that one can go from ω to ϕ by local operations and vice-versa, then $E(\omega) = E(\phi)$. Property 5 states the behavior of the entanglement when the subsystems \mathfrak{A}_1 and \mathfrak{A}_2 are themselves composed of statistically independent subsystems.

There are many examples of entanglement measures in the literature [55]. However, not all of them satisfy all the properties above but a proper subset. In particular, we can mention the *relative entanglement entropy*

$$E_R(\omega) := \inf \{S(\omega | \phi) : \phi \text{ is separable}\} , \quad (3.82)$$

which satisfies properties 1 to 4, and the mutual information

$$E_I(\omega) := I(\mathfrak{A}_1, \mathfrak{A}_2; \omega) , \quad (3.83)$$

which satisfies properties 1, 3 and 4, but it satisfies 2 only over the subset of pure states. We always have $E_R(\omega) \leq E_I(\omega)$ [55].

Chapter 4

Entanglement in QFT

In this chapter, we study general aspects of entanglement in QFT. For this purpose, we start with a general explanation about how the previous discussion of entanglement in bipartite quantum systems fits in AQFT. In particular, we remark that the MI for two strictly spacelike separated regions is always well-defined in QFT. We discuss two famous theorems concerning entanglement in QFT. The first one is the Bisognano-Wichmann theorem, which gives a direct connection between modular evolution and spacetime symmetries in QFT. The second one is the “entanglement interpretation” of the Reeh-Schlieder theorem, which suggests that the vacuum state of any QFT is entangled at any distance.

Then we study the entanglement entropy (EE) in QFT. It is a quantity that measures the entanglement, on a given state, between the degrees of freedom of a region \mathcal{O} with the degrees of freedom of its commutant \mathcal{O}' . We show that it is an ill-defined quantity due to UV-divergences. These divergencies have their origin in the fact that QFT is a theory with a “continuum” of degrees of freedom. Then, to study EE in QFT we must regularize the theory placing a short distance cutoff. In the regularized theory, we see that the divergent behavior of the EE with the cutoff has a very rigid structure. In fact, the understanding of such a divergence structure, allows us to define a universal regularized EE and a minimally subtracted EE using the MI. These entanglement measures have all the properties expected for the EE, except for the positivity. Furthermore, they contain all the relevant and universal information about the entanglement between the region \mathcal{O} and \mathcal{O}' in the vacuum state.

In the rest of this chapter, we show how to compute the modular Hamiltonian and the EE for free fields using a lattice cutoff. In the presence of such a cutoff, the infinite type III vN algebras of QFT are transformed into type I vN algebras, and hence, all the formulas for finite quantum systems of the previous chapter apply there. In particular, we focus on Gaussian states. Such states are characterized by the fact that all their information is encoded in their two-point correlators. For these states, we obtain simple formulas for the modular Hamiltonian and the EE. In the end, we explain how all the expressions obtained

for the lattice model have to be interpreted in the continuum limit. All the expressions derived there are very useful for computational purposes. We use them in chapter 6 where we compute the modular Hamiltonian and the EE for the free chiral current field.

4.1. General structure of entanglement in AQFT

From the algebraic perspective, the study of the entanglement in QFT is as follows. As we explained in chapter 2, we start with a net of local C^* -algebras $\mathfrak{A}(\mathcal{O})$ indexed by the set of causally complete regions \mathcal{K} of the spacetime. Given two regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ we want to characterize the (quantum) correlations between the degrees of freedom localized in \mathcal{O}_1 and \mathcal{O}_2 . According to the discussion in the previous chapter, it must be done with the help of the algebras $\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2) \subset \mathfrak{A}$. If we want to study the entanglement between these two algebras on a state $\omega \in \mathfrak{S}(\mathfrak{A})$, it is useful to consider the GNS-representation $\pi_\omega : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ of ω and define the net of vN algebras $\mathcal{A}(\mathcal{O}) := \pi_\omega(\mathfrak{A}(\mathcal{O}))'' \subset \mathcal{B}(\mathcal{H}_\omega)$ with $\mathcal{O} \in \mathcal{K}$. This net of vN algebras allows us to study the entanglement across any two regions, not only for the state ω but for any π_ω -normal state ϕ (any state which is represented by a density matrix in such a representation). Moreover, any global normal state is *locally normal*, i.e. its restriction to any local algebra $\mathcal{A}(\mathcal{O})$ is normal. Then, if we assume that a local algebra $\mathcal{A}(\mathcal{O}_1)$ is in standard form, any normal state ϕ has a vector representative $|\Phi\rangle \in \mathcal{H}_\omega$, which is also a vector representative for any subalgebra $\mathcal{A}(\mathcal{O}_2)$ of any region $\mathcal{O}_2 \subset \mathcal{O}_1$. This picture fits perfectly when one considers the vacuum state ω_0 . For the vacuum representation $\pi_0 : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_0)$, due to Reeh-Schlieder theorem 2.62, the vacuum vector $|0\rangle$ is standard for any local algebra $\mathcal{A}(\mathcal{O})$ whenever $\mathcal{O} \in \mathcal{K}$ and $\mathcal{O}' \neq \emptyset$. From now on, we always work in the vacuum representation, unless it is otherwise stated.

Now given two local algebras $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2) \subset \mathcal{B}(\mathcal{H}_0)$, in order to have a well-defined entanglement structure, we need this pair of algebras to be *kinematically independent*, i.e. they commute.¹ It can be easily satisfied considering spacelike separated regions $\mathcal{O}_1 \times \mathcal{O}_2$. In particular, it includes the case when $\mathcal{O}_2 = \mathcal{O}'_1$, which is frequently used in QFT. We also assume that $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ do not have a non-trivial element in common, i.e. $\mathcal{A}(\mathcal{O}_1) \cap \mathcal{A}(\mathcal{O}_2) = \mathbf{1}$. This is automatically satisfied whenever $\mathcal{A}(\mathcal{O}_1)$ or $\mathcal{A}(\mathcal{O}_2)$ is a factor, and this holds under the assumption that the vacuum representation is irreducible, i.e. $\mathcal{A}(\mathbb{R}^d)' = \mathbf{1}$ (see discussion in section 2.2.5).

We also require that the algebras $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ are *statistically independent (in the spatial product sense)* [92], i.e.

$$\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2) \cong \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2) , \quad (4.1)$$

¹For fermion nets, we impose twisted commutativity (see section 2.2.4).

where \cong means unitarily equivalence. We remark that the vN algebra on the l.h.s. of (4.1) acts on \mathcal{H}_0 , whereas the algebra on the r.h.s. of (4.1) acts on $\mathcal{H}_0 \otimes \mathcal{H}_0$. If we assume additivity for this pair of regions, we also have that

$$\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) = \mathcal{A}(\mathcal{O}_1 \vee \mathcal{O}_2) , \quad (4.2)$$

where the last equality holds since \mathcal{O}_1 and \mathcal{O}_2 are spacelike separated. The statistical independence relation (4.1) is a non-trivial property which may not hold in general. It could be derived from a stronger assumption called split-property [92].

Definition 4.1. Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{B}(\mathcal{H})$ be two commuting vN algebras. We say that the pair $\mathcal{A}_1, \mathcal{A}_2$ satisfies the *split-property* if there exists a type I factor \mathcal{R} such that $\mathcal{A}_1 \subset \mathcal{R} \subset \mathcal{A}'_2$.

Roughly speaking, in QFT, the split property for local algebras attached to strictly spacelike separated regions $\mathcal{O}_1 \times \mathcal{O}_2$ is a condition about the number of local degrees of freedom. It is well-known that the number of degrees of freedom of a local algebra is infinite, but such infinity is not so “large”. For example, a QFT defined by a set of independent local fields $\phi_i(x)$ does not satisfy the split property if the number of such independent fields is infinite. On the other hand, Buchholz has proven that free field theories satisfy the split property for strictly separated regions [93]. Moreover, Buchholz and Wichmann argue that the split property for strictly separated spacelike regions is a sufficient condition for a particle interpretation of QFT [94]. For our purposes, the split-property is an extra assumption.

Assumption 4.2. *The local algebras $\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)$ of any pair of strictly separated spacelike regions $\mathcal{O}_1 \times \mathcal{O}_2$ satisfies the split-property. Hence, (4.1) holds.*

Remark 4.3. It is important to emphasize that the above property only holds for algebras attached to strictly separated regions. It is a well-known fact that, even for free fields, the algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O}')$ do not satisfy (4.1), and hence they do not satisfy the split-property. This means that the entanglement structure of complementary regions is not completely well-defined. From the mathematical point of view, this could be understood as a reason why the entanglement entropy diverges in QFT.

Since relation (4.1) means unitarily equivalence, we have that any normal ω on $\mathcal{A}(\mathcal{O}_1 \vee \mathcal{O}_2)$ is mapped to a normal state $\tilde{\omega}$ on $\mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)$. Moreover, the expectation values of both states in their corresponding algebras are equivalent, i.e. $\omega(A_1 A_2) = \tilde{\omega}(A_1 \otimes A_2)$. We will denote both states simply as ω .

In conclusion, we are able to map our original system to a bipartite quantum system. Now, we can compute any entanglement measure as we have explained in the previous chapter. As an example, the mutual information for any normal state ω on $\mathcal{A}(\mathcal{O}_1 \vee \mathcal{O}_2)$ is

$$I(\mathcal{O}_1, \mathcal{O}_2; \omega) := I(\mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2); \omega) , \quad (4.3)$$

where the r.h.s. is computed using definition 3.60. It will be denoted by $I(\mathcal{O}_1, \mathcal{O}_2)$ or $I(\omega)$ whenever there is no confusion about the state or the algebras involved. Then, the mutual information is a well-defined quantity in QFT for two strictly separated regions. Moreover, if the underlying state is the vacuum ω_0 , it can be shown that it is also finite. For the relative entropy, the above discussion is even simpler. It is because, in this case, we deal with only one algebra $\mathcal{A}(\mathcal{O})$ and there is no tensor product structure. Then, for any two normal states ω, ϕ we define

$$S_{\mathcal{O}}(\phi | \omega) := S_{\mathcal{A}(\mathcal{O})}(\phi | \omega) . \quad (4.4)$$

Remark 4.4. In the case of fermionic nets, expression (4.1) can never hold because of the presence of fermionic operators anticommuting at spacelike distances. However, in that case, relation (4.1) still holds if the tensor product is replaced by the *graded tensor product*. Then, the mutual information for fermionic nets is also well-defined [62].

In conclusion, MI and RE are well-defined entanglement quantities in QFT. However, as we have explained in the previous chapter, the entanglement entropy (EE) is not well-defined since the local algebras of QFT are type III vN algebras. At this point we could take two paths: a) discard completely EE in QFT, or b) try to understand its divergent structure and extract the “finite” relevant information about the entanglement from such a divergent quantity. Of course, here we follow b). In the end, we define a “universal” regularized EE which works for any QFT.

4.1.1. Bisognano-Wichmann theorem

The study of modular Hamiltonians in QFT is a very intriguing problem, both for the study of entanglement or just for merely algebraic interests. In any QFT, we have the distinguished vacuum state ω_0 , which is represented by the vector $|0\rangle$. Hence, an interesting task is to compute the modular Hamiltonian for such a vector and any local algebra $\mathcal{A}(\mathcal{O})$. However, this problem is far from being solved in general, except for some particular examples, which in general, involve free field theories. Despite this, there is a particular kind of regions, called wedges, for which the modular flow of the vacuum vector has a “geometrical action”. This is the very famous theorem of Bisognano-Wichmann, which we explain in this section.

Let first define the wedge regions. The prototypical example of a wedge region is the *right Rindler wedge*

$$\mathcal{W}_R := \{x \in \mathbb{R}^d : x^1 > |x^0|\} . \quad (4.5)$$

A general *wedge region* is defined to be any region \mathcal{W} obtained from \mathcal{W}_R applying a general Poincaré transformation, i.e. $\mathcal{W} = g\mathcal{W}_R$ with $g \in \mathcal{P}_+^\uparrow$. Throughout this section, it

is convenient to write any spacetime point as $x = (x^0, x^1, \bar{x}_\perp)$, where $\bar{x}_\perp := (x^2, \dots, x^{d-1})$ are called the “orthogonal coordinates” with respect to the wedge \mathcal{W}_R . Now we state the Bisognano-Wichmann theorem.

Theorem 4.5. (*Bisognano-Wichmann*) *In the vacuum representation of any QFT, the modular operator Δ_0 and the modular conjugation J_0 associated with $\{\mathcal{A}(\mathcal{W}_R), |0\rangle\}$ are*

$$J_0 = \Theta \quad \text{and} \quad \Delta_0 = e^{-2\pi K_1}, \quad (4.6)$$

where Θ is the CRT operator² and K_1 is the infinitesimal generator of the one-parameter group of boost symmetries in the plane (x^0, x^1) , i.e.

$$U(\Lambda_1^s, 0) = e^{iK_1 s}, \quad \text{with } \Lambda_1^s := \begin{pmatrix} \cosh(s) & \sinh(s) & \mathbf{0} \\ \sinh(s) & \cosh(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}. \quad (4.7)$$

Furthermore, wedge duality $\mathcal{A}(\mathcal{W}'_R) = \mathcal{A}(\mathcal{W}_R)'$ holds.

Bisognano and Wichmann were the first who stated and proved this theorem for the theory of a real scalar field [33]. However, this theorem holds for any AQFT in the vacuum representation [59]. Moreover, it holds for any wedge region due to the Poincaré invariance of the theory. In fact, for any other wedge $\mathcal{W} = g\mathcal{W}_R$, the modular operator $\Delta_{\mathcal{W}}$ and the modular conjugation $J_{\mathcal{W}}$ associated with $\{\mathcal{A}(\mathcal{W}), |0\rangle\}$ are $J_{\mathcal{W}} = U(g) J_0 U(g)^*$ and $\Delta_{\mathcal{W}} = U(g) \Delta_0 U(g)^*$.

From (4.6) we easily get the relation $K_0 = 2\pi K_1$, which could be written as

$$K_0 = 2\pi \int_{\mathbb{R}^d} d^{d-1}x x^1 T_{00}(x), \quad (4.8)$$

whenever the theory has a stress-tensor field operator. Inspired by the discussion in remark 3.15 and example 3.18, one is tempted to write the following expression for the “inner” modular Hamiltonian

$$K_0^R = 2\pi \int_{x^1 > 0} d^{d-1}x x^1 T_{00}(x), \quad (4.9)$$

where the super-index R comes from “right” and indicates that it is the “half” of K_0 belonging to the right Wedge region. The above formula is heuristically correct because, for any operator $A \in \mathcal{A}(\mathcal{W}_R)$, the modular flow could be written as $\sigma_t(A) \equiv e^{-i2\pi K_0^R t} A e^{i2\pi K_0^R t}$. However, expression (4.9) is not completely correct because it is mathematically ill-defined. It can be shown that K_0^R is not an operator,³ and in particular, we have that $\|K_0^R |0\rangle\| =$

²The CRT operator is the generalization of the CPT operator to any spacetime dimensions d . In $d = 4$, CRT and CPT differ just by a spatial rotation.

³In the sense as a linear transformation on the Hilbert space with dense domain.

$+\infty$. Moreover, the full modular Hamiltonian can never be decomposed as $K_0 = K_0^R - K_0^{R'}$, because the algebras $\mathcal{A}(\mathcal{W}_R)$ and $\mathcal{A}(\mathcal{W}'_R)$ are not statistically independent in the sense of (4.1). However, K_0^R is well-defined as a sesquilinear form: expectation values $\langle \Psi_1 | K_0^R | \Psi_2 \rangle$ are finite for all $|\Psi_j\rangle$ in a dense subset of the Hilbert space.

Two remarkable things follow from the above theorem. The first one is that for algebras associated with wedge regions, the modular operator is universal, i.e. it does not depend on the theory. The second one is that the modular flow $A \rightarrow \sigma_t(A) = \Delta_0^{it} A \Delta_0^{-it}$ has a geometrical action on the algebra $\mathcal{A}(\mathcal{W}_R)$, in the sense that if $A \in \mathcal{A}(\mathcal{O})$ with $\mathcal{O} \subset \mathcal{W}_R$, then $\sigma_t(A) \in \mathcal{A}(U(\Lambda_1^s, 0)\mathcal{O})$ with $s = -2\pi t$. Or equivalently, if the theory is defined by means a field operator $\phi(x)$, then $\sigma_t(\phi(x)) = \phi(\Lambda_1^{-2\pi t}x)$ for all $x \in \mathcal{W}_R$ and all $t \in \mathbb{R}$. In other words, the modular evolution is given by the boost evolution, which is a spacetime symmetry. This gives a relation between spacetime and modular theory. Unfortunately, such behavior does not generalize to any other kind of region. The modular group associated with a local algebra $\mathcal{A}(\mathcal{O})$ of a general region \mathcal{O} other than a wedge, gives an automorphism of the algebra, which cannot be geometrical [37]. There is an exception, which is the double cones in CFTs. In those cases, the modular flow is also universal (independent of the theory), and it is given by the one-parameter group of conformal transformations which leaves the double cone invariant [34, 35].

The above interesting relation between spacetime symmetries and modular theory can be reversed. Let us start with a net of vN algebras associated with wedges, i.e. an assignment to each wedge region \mathcal{W} a vN algebra $\mathcal{W} \mapsto \mathcal{A}(\mathcal{W}) \subset \mathcal{B}(\mathcal{H}_0)$. We assume that this net satisfies the three first axioms of definition 2.56, and we also assume that all the wedge algebras $\mathcal{A}(\mathcal{W})$ have a common standard vector $|0\rangle \in \mathcal{H}$. Let us denote by $\Delta_{\mathcal{W}}^{it}$ the modular group associated with $\{\mathcal{A}(\mathcal{W}), |0\rangle\}$. Instead of assuming the Poincaré covariance of the net (axiom 4 of definition 2.56), we assume the following condition.

4'. (*Bisognano-Wichmann property*) For any wedges $\mathcal{W}_1, \mathcal{W}_2$, we have that

$$\Delta_{\mathcal{W}_1}^{it} \mathcal{A}(\mathcal{W}_2) \Delta_{\mathcal{W}_1}^{-it} = \mathcal{A}(\Lambda_{\mathcal{W}_1}^{-2\pi t} \mathcal{W}_2), \quad (4.10)$$

where $\Lambda_{\mathcal{W}_1}^{-2\pi t}$ is the one-parameter group of Lorentz matrices leaving the wedge \mathcal{W}_1 invariant.

This condition means that the wedge algebras transform accordingly to the statement of theorem 4.5. Under the above assumptions, it can be shown that there exists a unique unitary representation of the Poincaré group $g \in \mathcal{P}_+^\uparrow \mapsto U(g) \in \mathcal{B}(\mathcal{H}_0)$ such that

$$\Delta_{\mathcal{W}}^{it} = U(\Lambda_{\mathcal{W}}^{-2\pi t}), \quad (4.11)$$

$$\mathcal{A}(g\mathcal{W}) = U(g) \mathcal{A}(\mathcal{W}) U(g)^*, \quad (4.12)$$

for all $g \in \mathcal{P}_+^\dagger$, all $t \in \mathbb{R}$ and all wedges \mathcal{W} [95]. In other words, the quantum Poincaré symmetry can be reconstructed assuming the Bisognano-Wichmann property. This leaves open the question if the Bisognano-Wichmann property could be a “more fundamental” condition than the Poincaré symmetry.

4.1.2. Reeh-Schlieder theorem

In this subsection, we show that Reeh-Schlieder theorem 2.62 admits an interpretation in terms of entanglement. For such a purpose, we make use of an example concerning finite dimensional algebras. In section 3.5, we have introduced the bell pair

$$|\Psi\rangle := \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) , \quad (4.13)$$

as a maximally entangled state for the bipartite system $\mathfrak{A} := M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. such a vector has the interesting property that, acting on it with operators belonging to one of the subsystems, we can construct any vector of the full Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. More precisely, any single qubit operator

$$A := \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \in M_2(\mathbb{C}) , \quad (4.14)$$

acts over the basis vectors as

$$A|0\rangle = a_{00}|0\rangle + a_{01}|1\rangle , \quad A|1\rangle = a_{10}|0\rangle + a_{11}|1\rangle . \quad (4.15)$$

Then, acting with such an operator on (4.13) we have that

$$(A \otimes \mathbf{1})|\Psi\rangle = \frac{1}{\sqrt{2}} (a_{00}|0\rangle \otimes |0\rangle + a_{01}|1\rangle \otimes |0\rangle + a_{10}|0\rangle \otimes |1\rangle + a_{11}|1\rangle \otimes |1\rangle) . \quad (4.16)$$

In other words, choosing the operator A conveniently, we can reach any vector of the full Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. This behavior is only possible because of the particular structure of the vector (4.13). Moreover, it also works for any global vector state whose reduced state, to any of the subsystems, is faithful, i.e. the reduced density matrices are invertible. In contrast, the above fact does not hold for the separable vector $|0\rangle \otimes |0\rangle$.

The similarity with the vacuum vector $|0\rangle$ in QFT is evident. Given a region \mathcal{O} , the full observable algebra can be decomposed as

$$\mathcal{B}(\mathcal{H}_0) = \mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\mathcal{O}') . \quad (4.17)$$

The Reeh-Schlieder theorem asserts that acting with operators $A \in \mathcal{A}(\mathcal{O})$ on the vacuum

vector $|0\rangle$ we can reach (almost⁴) any vector in \mathcal{H}_0 . Following the example above, we say that the vacuum state is entangled for the pair $\{\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}')\}$. Moreover, since it happens for any region \mathcal{O} , we say that the vacuum vector is entangled at any distance. Against intuition, the vacuum is a highly entangled state [96].

The Reeh-Schlieder theorem may seem paradoxical at first. It implies that by acting on the vacuum with an operator supported in a small region \mathcal{O} , one can create whatever one wants. Indeed, for any two spacelike separated regions $\mathcal{O}_1 \times \mathcal{O}_2$ and any operator $A_1 \in \mathcal{A}(\mathcal{O}_1)$, the vector $A_1 |0\rangle$ can always be reproduced by the action of an operator $A_2 \in \mathcal{A}(\mathcal{O}_2)$ on $|0\rangle$, regardless how far is \mathcal{O}_1 from \mathcal{O}_2 . However, it is no longer true that performing unitary operators inside a small region \mathcal{O} , we can create whatever state we want supported in a far spacelike separated region. This is because in order that the Reeh-Schlieder theorem holds, we have to be allowed to act on the vacuum vector with any local operator. The Reeh-Schlieder theorem is no longer true for the restricted subset of unitary operators.

4.2. Entanglement entropy in QFT

As we have discussed in the previous section, the vacuum state ω_0 is a “highly” entangled state in QFT. The aim of this section is to quantify such entanglement between different regions of spacetime. More concretely, we consider a causally complete region $\mathcal{O} \in \mathcal{K}$ and the local algebra $\mathcal{A}(\mathcal{O})$. We look for a measure of entanglement, for the vacuum state, between $\mathcal{A}(\mathcal{O})$ and the algebra of its complementary region $\mathcal{A}(\mathcal{O}')$.

Given a causally complete region $\mathcal{O} \in \mathcal{K}$, there always exists a (non-unique) Cauchy surface Σ and a spacelike region $\mathcal{C} \subset \Sigma$, such that \mathcal{O} is the Cauchy development of \mathcal{C} , i.e. $\mathcal{O} = D(\mathcal{C})$. The (spacelike codimension 2) boundary $\partial\mathcal{C}$ of \mathcal{C} inside Σ is called the *entanglement surface* between \mathcal{O} and \mathcal{O}' , and it is denoted by $\gamma_{\mathcal{O}}$ (see figure 4.1). It is not difficult to see that the entanglement surface is independent on the choice of Σ , and furthermore, $\gamma_{\mathcal{O}} = \gamma_{\mathcal{O}'}$ for any $\mathcal{O} \in \mathcal{K}$.

For a finite bipartite system, the entanglement entropy (3.75) is a good measurement of entanglement between the two subsystems for any pure state (section 3.5). However, as we have already discussed, we have the following two problems in QFT.

1. $\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\mathcal{O}') \not\cong \mathcal{A}(\mathcal{O}) \otimes \mathcal{A}(\mathcal{O}')$, and hence, the structure explained in section 3.5 does not apply in this case.
2. The local algebra $\mathcal{A}(\mathcal{O})$ is a type III vN algebra. This implies that the vN entropy of any state on such an algebra is ill-defined (section 3.3).

⁴The word “almost” is there because $\mathcal{A}(\mathcal{O}) |0\rangle$ is not all, but a dense subset of \mathcal{H}_0 .

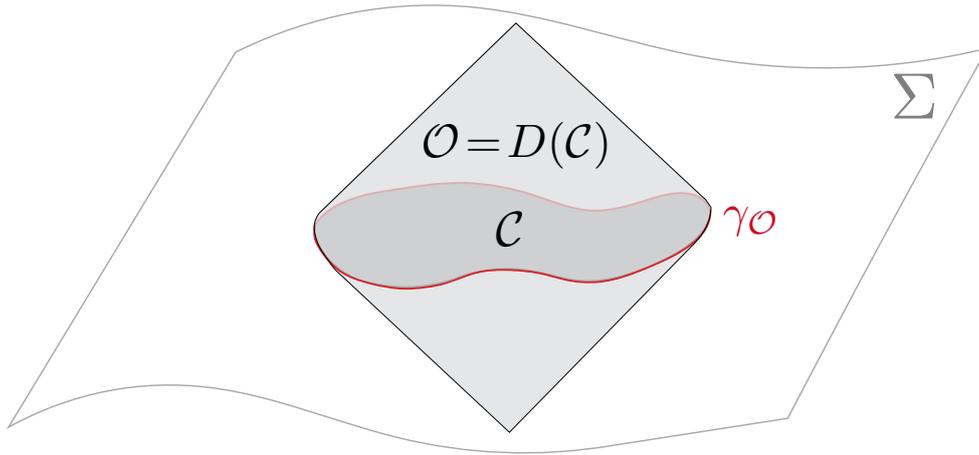


Figure 4.1: The causally complete region \mathcal{O} is the Cauchy development of the spacelike region \mathcal{C} , which lies inside the Cauchy surface Σ . The entanglement surface $\gamma_{\mathcal{O}}$ is the boundary $\partial\mathcal{C}$ of \mathcal{C} inside Σ . For a given causally complete region \mathcal{O} , there are many Cauchy surfaces and spacelike regions giving the same configuration. However, $\gamma_{\mathcal{O}}$ is independent of such choices.

In order to bypass such issues, we need to regularize the theory introducing a UV cutoff δ . For example, the lattice cutoff fits very well in the algebraic setting. In this case, the local algebras of QFT are represented by a lattice of (bosonic or fermionic) harmonic oscillators separated by a distance $\delta > 0$.⁵ Then, in the regularized QFT, the regularized local algebras $\mathcal{A}_{\delta}(\mathcal{O})$ are type I vN algebras and they satisfy

$$\mathcal{A}_{\delta}(\mathcal{O}) \vee \mathcal{A}_{\delta}(\mathcal{O}') \cong \mathcal{A}(\mathcal{O}) \otimes \mathcal{A}(\mathcal{O}') . \quad (4.18)$$

Hence, we can define the regularized EE as

$$S_{\delta}(\mathcal{O}) := -\text{Tr}_{\mathcal{A}_{\delta}(\mathcal{O})}(\rho_{\mathcal{O}} \log \rho_{\mathcal{O}}) , \quad (4.19)$$

where $\rho_{\mathcal{O}}$ is the statistical operator of the restricted state $\omega|_{\mathcal{A}_{\delta}(\mathcal{O})}$. However, in the continuum limit $\delta \rightarrow 0^+$, we always get $S_{\delta}(\mathcal{O}) \rightarrow +\infty$. It is for this reason that we say that the EE is UV divergent in QFT.

The above behavior has nothing to do with the special choice of the UV lattice cutoff. Indeed, it can be shown, using different techniques and cutoffs, that the EE always diverges in the continuum limit $\delta \rightarrow 0^+$, even for free theories. Furthermore, such behavior could not be remedied without spoiling some of the distinct properties of the EE. To be more precise, we must remark which are these properties.

Definition 4.6. In QFT, the EE for the vacuum state is a function from the set of causally complete regions into the real numbers, i.e. $\mathcal{O} \in \mathcal{K} \mapsto S(\mathcal{O}) \in \mathbb{R}$, which satisfies the following properties.

⁵We explain this in detail in section 4.3.

1. (positivity) $S(\mathcal{O}) \geq 0$ for all $\mathcal{O} \in \mathcal{K}$.
2. (strong subadditivity) $S(\mathcal{O}_1 \vee \mathcal{O}_2) + S(\mathcal{O}_1 \wedge \mathcal{O}_2) \leq S(\mathcal{O}_1) + S(\mathcal{O}_2)$ for all commuting⁶ regions \mathcal{O}_1 and \mathcal{O}_2 .
3. (Poincaré invariance) $S(g\mathcal{O}) = S(\mathcal{O})$ for all $g \in \mathcal{P}_+^\uparrow$.

The first two conditions are motivated by the definition of the EE for finite quantum systems. The third condition is stated because the vacuum state is Poincaré invariant, and the local algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(g\mathcal{O})$ are unitarily equivalent because of the Poincaré covariance of the theory. However, there is a lemma that asserts that a non-trivial EE cannot exist in QFT.

Lemma 4.7. *Let $\mathcal{K}_p \subset \mathcal{K}$ denotes the subset of causally complete regions with polyhedral entanglement surface. Any positive, strong subadditive and Poincaré invariant function $\mathcal{O} \in \mathcal{K}_p \mapsto S(\mathcal{O}) \in \mathbb{R}$ is of the form*

$$S(\mathcal{O}) = c_1 + c_2 \text{vol}(\gamma_{\mathcal{O}}), \quad (4.20)$$

where $\text{vol}(\gamma_{\mathcal{O}})$ is the geometrical volume of the (spacelike codimension 2) entanglement surface $\gamma_{\mathcal{O}}$ and $c_1, c_2 \geq 0$ are independent of the region \mathcal{O} .

Proof. See [97]. □

Assuming that (4.20) represents the EE, then we would have for the MI

$$I(\mathcal{O}_1, \mathcal{O}_2) = S(\mathcal{O}_1) + S(\mathcal{O}_2) - S(\mathcal{O}_1 \cup \mathcal{O}_2) = c_1, \quad (4.21)$$

for all strictly spacelike separated regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_p$. Furthermore, if we assume that the theory satisfies clustering, the correlations and hence the MI, between \mathcal{O}_1 and \mathcal{O}_2 would vanish whenever the separation between the regions goes to spacelike infinity. This would imply that $c_1 = 0$, and therefore, $I(\mathcal{O}_1, \mathcal{O}_2) = 0$ for all strictly spacelike separated regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_p$. Moreover, given any two strictly spacelike separated regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$, there always exist two strictly spacelike separated regions $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2 \in \mathcal{K}_p$ such that $\tilde{\mathcal{O}}_j \supset \mathcal{O}_j$. Then, by monotonicity of the MI, we would have that $I(\mathcal{O}_1, \mathcal{O}_2) = 0$ for all strictly spacelike separated regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$.

This means that (4.20) is not a good entanglement measure since, as we have discussed in section 4.1.2, the vacuum state is always entangled at any distance. Regularized QFTs have a well-defined EE at the expense of giving up some of the properties of definition 4.6. For example, the EE of a lattice QFT does not satisfy property 3, because Poincaré covariance is explicitly broken by the regularization.

⁶For the definition of commuting regions see equation (2.52) and definition A.16 in appendix A.

In conclusion, the EE is always UV divergent in QFT. However, its divergence structure has a very rigid form. We will discuss this issue in the following subsection.

4.2.1. Divergence structure of the EE

In the presence of a UV cutoff δ with mass dimension -1 , the leading divergent contribution to the EE grows with the volume of the entanglement surface [15–17]

$$S_\delta(\mathcal{O}) = \frac{\tilde{\mu}_{d-2}}{\delta^{d-2}} \text{vol}(\gamma_{\mathcal{O}}) + \text{non-leading terms} , \quad (4.22)$$

which is commonly known as *area law*. The coefficient $\tilde{\mu}_{d-2}$ is a dimensionless constant that depends on the theory and the regularization prescription. Physically, this divergence is a consequence of the existence, in QFT, of an infinite number of degrees of freedom which are entangled in the vacuum state. Because the vacuum state is the ground state of a local Hamiltonian, the degrees of freedom which contribute more to the entanglement between \mathcal{O} and its complement \mathcal{O}' are those located arbitrarily close and at opposite sides of the entanglement surface. Due to the same locality of the short distance correlations, the degrees of freedom which produce the UV divergences are disposed extensively along the entanglement surface $\gamma_{\mathcal{O}}$ (see figure 4.2). This fact allows us to better understand the structure of divergencies of $S_\delta(\mathcal{O})$, which is generically given by a sum of terms that are local and extensive along the entangling surface $\gamma_{\mathcal{O}}$.

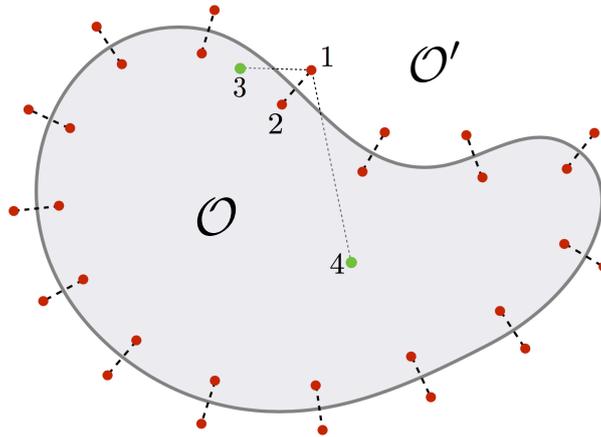


Figure 4.2: In QFT, the main contribution to the EE of the vacuum state restricted to a region \mathcal{O} is given by the entanglement between the degrees of freedom which are arbitrarily close and at opposite sides of the entanglement surface $\gamma_{\mathcal{O}}$. This contribution is extensive along $\gamma_{\mathcal{O}}$. Heuristically, because of the locality of the Hamiltonian, the entanglement between the degree of freedom located in the point 1 with the one located in the point 3 or in the point 4, is always less than the entanglement with the degree of freedom in the point 2, which is the closest degree of freedom (belonging to \mathcal{O}) to the one at point 1. As a consequence, the divergent part of the EE is local and extensive along $\gamma_{\mathcal{O}}$.

From now on, we fix a specific CFT and we take the vacuum state. The aim is to characterize the behavior of $S_\delta(\mathcal{O})$ in terms of \mathcal{O} for a fixed regularization. For the moment

we just consider regions \mathcal{O} with smooth entanglement surface $\gamma_{\mathcal{O}}$. Since in a CFT there is no dimensionful parameter, we expect the following expansion for the regularized EE [98]

$$S_{\delta}(\mathcal{O}) = \frac{\mu_{d-2}(\gamma_{\mathcal{O}})}{\delta^{d-2}} + \frac{\mu_{d-4}(\gamma_{\mathcal{O}})}{\delta^{d-4}} + \frac{\mu_{d-6}(\gamma_{\mathcal{O}})}{\delta^{d-6}} + \dots + S_0(\mathcal{O}) + \text{positive powers of } \delta. \quad (4.23)$$

The above expansion is in inverse powers of the cutoff δ . The parameters $\mu_{d-2k}(\gamma_{\mathcal{O}})$ are local and extensive along the entanglement surface $\gamma_{\mathcal{O}}$. Local means that they depend on the geometry of the region and in the cutoff UV behavior close to the entanglement surface. Extensive means that the contributions are additive along $\gamma_{\mathcal{O}}$, and hence, they can be written as integrals along $\gamma_{\mathcal{O}}$. The mass dimension of $\mu_{d-2k}(\gamma_{\mathcal{O}})$ must be $2k - d$ because the EE is dimensionless. The coefficient of leading term is $\mu_{d-2}(\gamma_{\mathcal{O}}) = \tilde{\mu}_{d-2} \text{vol}(\gamma_{\mathcal{O}})$ with $\tilde{\mu}_{d-2}$ dimensionless, which gives the area law as we have explained above. For a general regularization, the parameters $\mu_{d-2k}(\gamma_{\mathcal{O}})$ have extensive contributions coming from the (intrinsic and extrinsic) geometry of the entanglement surface $\gamma_{\mathcal{O}}$, but they also have extra local and extensive contributions constructed with parameters coming from the regularization. These quantities are *non-universal* in the sense that they depend on the regularization scheme used.

Among the different regularization schemes, there are the so-called “geometric regularizations”. Such regulators are characterized by the fact that the quantities $\mu_{d-2k}(\gamma_{\mathcal{O}})$ are purely geometrical in the sense that they can be written as

$$\mu_{d-2k}(\gamma_{\mathcal{O}}) = \int_{\gamma_{\mathcal{O}}} d\sigma \sqrt{h} R_{d-2k}[h_{ab}, K_{ab}^{\nu}], \quad (4.24)$$

where $R_k[h_{ab}, K_{ab}^{\nu}]$ represents a generic curvature scalar, constructed from the induced metric h_{ab} and the extrinsic curvatures K_{ab}^{ν} ($\nu = 1, 2$ runs over two normal directions) of the surface $\gamma_{\mathcal{O}}$, and $\sqrt{h}d\sigma$ is the usual volume element over the surface $\gamma_{\mathcal{O}}$. For a geometrical cutoff, it is expected that the expansion (4.23) is perturbative in the geometry of $\gamma_{\mathcal{O}}$. This means that the geometrical integrands of (4.24) can only involve positive integer powers of the curvature scalars R_{d-2k} . Then, because the cutoff is the unique mass scale in the theory, only integer powers of the cutoff can appear in the subleading terms of (4.22), justifying the ansatz (4.23). The coefficients $\mu_{d-2k}(\gamma_{\mathcal{O}})$ are also non-universal quantities, i.e. they are regularization dependent. However, in section 4.2.3, we show a “universal” way to choose a geometrical regularization prescription for any QFT. From now on, we only consider geometric regularizations.

We remark that the expansion (4.23) does not involve terms proportional to δ^{d-l} with $l > 0$ and odd. These terms, which could be present in an arbitrary regularization of the

theory, will involve integrals of geometric quantities of even mass dimensions. Any such quantity must contain odd powers of the extrinsic curvatures K_{ab}^ν of $\gamma_{\mathcal{O}}$, and therefore, contribute with opposite sign for \mathcal{O} and \mathcal{O}' . However, in a regularization where such terms are present, the duality $S_\delta(\mathcal{O}) = S_\delta(\mathcal{O}')$, which is expected whenever the global state is pure, fails.⁷ This is because such a contribution has opposite signs when is calculated for \mathcal{O} and \mathcal{O}' . Hence, whenever we write (4.23), we are implicitly assuming a regularization scheme that preserves the mentioned duality.

The constant term $S_0(\mathcal{O})$ goes as $\sim \delta^0$ for odd d dimensions and goes as $\sim \log(\delta)$ for even d dimensions. In fact, we expect

$$S_0(\mathcal{O}) = \begin{cases} (-1)^{\frac{d-1}{2}} s_0(\mathcal{O}), & \text{odd } d, \\ (-1)^{\frac{d-2}{2}} s_0(\mathcal{O}) \log\left(\frac{R_{\mathcal{O}}}{\delta}\right) + \tilde{s}_0(\mathcal{O}), & \text{even } d, \end{cases} \quad (4.25)$$

where $R_{\mathcal{O}}$ is any length parameter of the region \mathcal{O} , which scales as $R_{\mathcal{O}} \rightarrow \lambda R_{\mathcal{O}}$ for dilations $x \rightarrow \lambda x$. The sign factors before $s_0(\mathcal{O})$ are chosen for later convenience. $s_0(\mathcal{O})$ and $\tilde{s}_0(\mathcal{O})$ are non-local quantities, and they depend only on the shape but not on the size of the region \mathcal{O} . We are not usually interested in the terms with positive powers of δ because they vanish in the limit of $\delta \rightarrow 0^+$.

The important fact is that the term $S_0(\mathcal{O})$ contains valuable information about the entanglement between \mathcal{O} and \mathcal{O}' . In fact, under the assumptions above, $s_0(\mathcal{O})$ are universal quantities belonging to the continuum CFT. For example, when \mathcal{O} is the Cauchy development of a $d-1$ dimensional sphere of radius R , we have that

$$S_0(\mathcal{O}) = \begin{cases} (-1)^{\frac{d-1}{2}} F, & \text{odd } d, \\ (-1)^{\frac{d-2}{2}} 4A \log\left(\frac{R}{\delta}\right) + \text{const}, & \text{even } d, \end{cases} \quad (4.26)$$

where A is the coefficient of the Euler density in the trace anomaly [99] and F is the constant term of the free energy in a d dimensional Euclidean sphere [35, 100, 101]. These coefficients are precisely the quantities that decrease under renormalization group flow in the c , F and A theorems in dimension $d = 2, 3, 4$ respectively [26–32]. For even d , the universal part of the term $\tilde{s}_0(\mathcal{O})$ is harder to be extracted, even in the case of spheres. This is because a change in the cutoff $\delta \rightarrow \lambda\delta$, or equivalently in the length parameter $R_{\mathcal{O}} \rightarrow \lambda^{-1}R_{\mathcal{O}}$, leads to a change in the finite part $\tilde{s}_0(\mathcal{O}) \rightarrow \tilde{s}_0(\mathcal{O}) - (-1)^{\frac{d-2}{2}} s_0(\mathcal{O}) \log(\lambda)$. To extract its universal part, we have to invoke relative quantities. For example, for two regions $\mathcal{O}_1 \otimes \mathcal{O}_2$ we can use the regularized mutual information

$$I_\delta(\mathcal{O}_1, \mathcal{O}_2) := S_\delta(\mathcal{O}_1) + S_\delta(\mathcal{O}_2) - S_\delta(\mathcal{O}_1 \vee \mathcal{O}_2). \quad (4.27)$$

⁷See section 4.2.2 for clarification.

In this case, the local and extensive contributions of the coefficient $\tilde{s}_0(\mathcal{O}_1) + \tilde{s}_0(\mathcal{O}_2) - \tilde{s}_0(\mathcal{O}_1 \vee \mathcal{O}_2)$ cancel out and it gives a universal quantity of the continuum CFT attached to the pair $\{\mathcal{O}_1, \mathcal{O}_2\}$.

The general expression (4.23) slightly changes when the entanglement surface is non-smooth. For example, it could be the case that $\gamma_{\mathcal{O}}$ has a defect like a corner. The local contributions to the entropy coming from points around such a defect are non-longer perturbative in the local geometry since, around there, the geometry does not look like flat geometry plus small smooth curvature deformations. In this case, there are extra divergent contributions attached to the defect [102–104].

In the case of free fields and for some simple geometries, the above structure of divergences could be explicitly verified [105]. This has also been verified in holographic theories using the Ryu-Takayanagi prescription.

In the case of a general QFT with non-conformal symmetry, the divergence structure of the EE is richer. The presence of dimensionful parameters on the theory enlarges the possible divergent terms. Now, let us consider a general QFT that is constructed by perturbing a CFT in the UV-fixed point by a relevant operator with mass dimension $\Delta \in (\frac{d-2}{2}, d)$

$$S = S_{CFT} + \lambda \int O_{\Delta}(x) d^d x, \quad (4.28)$$

and coupling constant λ with mass dimension $d - \Delta \in (0, \frac{d+2}{2})$. Then, the coefficients of the geometrical terms in the divergent part of the entropy are corrected. We could have terms of the form

$$\mu_{d-2k}(\gamma_{\mathcal{O}}) \frac{\lambda^n}{\delta^{d-2k-n(d-\Delta)}}, \quad n \in \mathbb{N}, n \neq 1. \quad (4.29)$$

It means that, besides the negative integer powers of the cutoff, there may appear new non-integer powers of the cutoff compensated with an adequate integer powers of the coupling constant. For precise values of Δ and $n \geq 2$, these new terms could be recombined with preexisting terms of less order in δ , modifying the value of the non-universal coefficients of $\mu_{d-2k}(\gamma_{\mathcal{O}})$. However, the presence of such terms cannot modify the leading divergent term proportional to δ^{-d+2} , which always grows with the area.

As we have emphasized, the divergent structure (4.23) is non-universal. In section 4.2.3, we introduce a general “geometric” way to regularize the EE, which holds for any QFT.

4.2.2. Equality of the EE for complementary regions

Here, we give a brief comment about the equality of the EE for complementary regions. As we explained in section 3.2, for any finite quantum system $\mathfrak{A} \subset M_n(\mathbb{C})$ and any global

pure state ω on the algebra $M_n(\mathbb{C})$, we have the following equality for the vN entropies

$$S(\mathfrak{A}) = S(\mathfrak{A}') , \quad (4.30)$$

where \mathfrak{A}' is the algebraic commutant of \mathfrak{A} inside $M_n(\mathbb{C})$. In QFT, we have a similar setup. We could ask ourselves whether for two complementary regions \mathcal{O} and \mathcal{O}' and any global pure state, we would have

$$S(\mathcal{O}) = S(\mathcal{O}') . \quad (4.31)$$

Unfortunately, such a question is meaningless because the EE is infinite in QFT. However, let us assume for a moment that the EE is finite and well-defined. According to relation (4.30), we expect the equality

$$S(\mathcal{A}(\mathcal{O})) = S(\mathcal{A}(\mathcal{O}')) , \quad (4.32)$$

which does not imply (4.31) unless

$$\mathcal{A}(\mathcal{O})' = \mathcal{A}(\mathcal{O}') . \quad (4.33)$$

In other words, the equality of the EE for complementary regions (on a global pure state) is a consequence of the duality condition (4.33) for the algebra $\mathcal{A}(\mathcal{O})$ attached to the region \mathcal{O} . However, in AQFT, it is not guaranteed that the duality (4.33) holds in general. In fact, it could be the case that such equality holds for some, but not all, regions. In chapters 6 and 7, we will study some models where the equality (4.33) holds for double cones, but not for regions made out of the union of two strictly spacelike separated double cones.

The ambiguous relation (4.31) could be unambiguously stated for a regularized EE (4.23). For example, in a case of a QFT regularized by a lattice cutoff δ , the regularized EE (4.23) satisfies the equality (4.31) whenever the regularized algebra $\mathcal{A}_\delta(\mathcal{O})$ satisfies the duality condition (4.33). For a QFT in the continuum, an equivalent and complete rigorous statement of the relation (4.31) could be stated using the relative entropy. In fact, for finite quantum systems, we can use the relations (3.24) and (3.25) for the algebras \mathfrak{A} and \mathfrak{A}' , and two global pure states $\omega(\cdot) = \langle \Omega | \cdot | \Omega \rangle$ and $\phi(\cdot) = \langle \Phi | \cdot | \Phi \rangle$

$$S_{\mathfrak{A}}(\phi | \omega) = \Delta \langle K_{\omega, \mathfrak{A}} \rangle - \Delta S_{\mathfrak{A}} , \quad (4.34)$$

$$S_{\mathfrak{A}'}(\phi | \omega) = \Delta \langle K_{\omega, \mathfrak{A}'} \rangle - \Delta S_{\mathfrak{A}'} . \quad (4.35)$$

Then, subtracting both expressions and using (4.30), we have that

$$S_{\mathfrak{A}}(\phi | \omega) - S_{\mathfrak{A}'}(\phi | \omega) = \Delta \langle K_{\omega, \mathfrak{A}} - K_{\omega, \mathfrak{A}'} \rangle . \quad (4.36)$$

According to remark 3.15 and example (3.18), $K_{\omega, \mathfrak{A}} - K_{\omega, \mathfrak{A}'}$ in the above expression is just

the “full” modular Hamiltonian associated with $\{\mathfrak{A}, |\Omega\rangle\}$, which will be also denoted by $K_{\omega, \mathfrak{A}}$. Therefore, equation (4.36) could be rewritten as

$$S_{\mathfrak{A}}(\phi | \omega) - S_{\mathfrak{A}'}(\phi | \omega) = \langle \Psi | K_{\omega, \mathfrak{A}} | \Psi \rangle = \phi(K_{\omega, \mathfrak{A}}) . \quad (4.37)$$

The interesting fact of the above relation, which encodes the information of equality (4.30), is that it involves relative entropies and a “full” modular Hamiltonian, which are objects well-defined in the continuum QFT. Moreover, the expression (4.37) could be directly proved using the tools of modular theory developed in section 3.3.1. In AQFT, relation (4.37) must be restated in the following way.

Lemma 4.8. *In any AQFT (in the vacuum representation) and for any two global pure states ω, ϕ on $\mathcal{B}(\mathcal{H}_0)$, given by the unit vectors $|\Omega\rangle, |\Phi\rangle$, we have the following equality*

$$S_{\mathcal{O}}(\phi | \omega) - S_{\mathcal{O}'}(\phi | \omega) = \phi(K_{\Omega}) , \quad (4.38)$$

whenever the duality relation $\mathcal{A}(\mathcal{O})' = \mathcal{A}(\mathcal{O}')$ holds. The operator K_{Ω} in (4.38) is the (full) modular Hamiltonian associated with $\{\mathcal{A}(\mathcal{O}), |\Omega\rangle\}$.

4.2.3. Universal regularized entanglement entropy

As we have explained in section 4.2, EE is necessarily divergent in QFT, and its structure of divergences is non-universal. Then, in order to obtain universal information out of it, we are forced to use relative entropy quantities. This is especially necessary, in the context of the present thesis, since we are discussing subtleties that could be hard to understand without this perspective.

We are going to use mutual information as a way to define a regularized entanglement entropy [100]. Or, more precisely, we are defining entropy in QFT as a quantity derived from mutual information. MI retains all the universal information of the EE and, as we have explained in section 4.1, it is a well-defined quantity for strictly separated regions, and it is always finite for the vacuum state.

The definition of EE from MI follows from the following observation. We take a global state ω and two regions $\mathcal{O}_1 \bowtie \mathcal{O}_2$. According to (3.27), it is expected that the mutual information $I(\mathcal{O}_1, \mathcal{O}_2)$ looks like

$$I(\mathcal{O}_1, \mathcal{O}_2) \text{ “ = ” } S(\mathcal{O}_1) + S(\mathcal{O}_2) - S(\mathcal{O}_1 \vee \mathcal{O}_2) . \quad (4.39)$$

The symbol “ = ” is placed to emphasize that such equality is just heuristic. In fact, the r.h.s. of (4.39) is ill-defined. The above relation must be considered as a heuristic motivation coming from finite quantum systems. However, if we place a short distance

cutoff $\delta > 0$, we are allowed to compute the bare entanglement entropies $S_\delta(\mathcal{O})$. In this case, equation (4.39) is no longer heuristic and could be restated as

$$I(\mathcal{O}_1, \mathcal{O}_2) = \lim_{\delta \rightarrow 0^+} [S_\delta(\mathcal{O}_1) + S_\delta(\mathcal{O}_2) - S_\delta(\mathcal{O}_1 \vee \mathcal{O}_2)]. \quad (4.40)$$

The above relation is independent of the chosen regularization prescription since local ambiguities due to the cutoff are local and extensive on the boundary, and they cancel in (4.39) as we take the continuum limit $\delta \rightarrow 0^+$.

Now we want to use $I(\mathcal{O}_1, \mathcal{O}_2)$, which is a function of two regions, to produce a quantity $S(\mathcal{O})$ that is a function of only one region \mathcal{O} , and it will be interpreted as a version of the EE suitable for any QFT. From now on, we assume that the global state ω is pure. We also consider the entanglement surface $\gamma_{\mathcal{O}}$ of the causally complete region \mathcal{O} . We take a spatial unit vector η exterior to $\gamma_{\mathcal{O}}$, and a short distance scale $\epsilon > 0$.⁸ We can construct two spatial surfaces, each on each side of $\gamma_{\mathcal{O}}$, using the elements of the “framing” (η, ϵ) , as

$$\gamma_{\mathcal{O}^+} = \gamma_{\mathcal{O}} + \frac{\epsilon}{2}\eta \quad \text{and} \quad \gamma_{\mathcal{O}^-} = \gamma_{\mathcal{O}} - \frac{\epsilon}{2}\eta. \quad (4.41)$$

We call $\mathcal{O}^- \subset \mathcal{O}$ the causally complete region with entanglement surface $\gamma_{\mathcal{O}^-}$, and $\mathcal{O}^+ \subset \mathcal{O}'$ the causally complete region with entanglement surface $\gamma_{\mathcal{O}^+}$ (see figure 4.3). Then, we use the mutual information $I(\mathcal{O}^+, \mathcal{O}^-)$ as a regularization of the entropy. More precisely, we define the *universal regularized entanglement entropy* as

$$S_{reg}(\mathcal{O}; \epsilon, \eta) := \frac{I(\mathcal{O}^+, \mathcal{O}^-)}{2}, \quad (4.42)$$

where in the second equality we have used (4.39). Definition (4.39) is justified by the heuristic relation (4.39). In fact, in the limit $\epsilon \rightarrow 0^+$ we expect that the r.h.s. of (4.39) looks like

$$S(\mathcal{O}^+) + S(\mathcal{O}^-) \rightarrow S(\mathcal{O}') + S(\mathcal{O}) = 2S(\mathcal{O}), \quad (4.43)$$

$$S(\mathcal{O}^+ \vee \mathcal{O}^-) \rightarrow S(\mathbb{R}^d) = 0. \quad (4.44)$$

These last relations were motivated by the equality of the vN entropy for complementary algebras which is always satisfied for a global pure state on finite algebras. We remark that we do not have to assume (4.33) to state (4.43) and (4.44). In fact, such formulas are just a guide to motivate the definition 4.42. $S_{reg}(\mathcal{O}; \epsilon, \eta)$ is a quantity that belongs to the continuum theory, and in particular, is Poincaré invariant for the vacuum state. The symmetric framing at both sides of $\gamma_{\mathcal{O}}$ in (4.41) gives equal regularized entropy for complementary regions, i.e. $S_{reg}(\mathcal{O}; \epsilon, \eta) = S_{reg}(\mathcal{O}'; \epsilon, -\eta)$. It is important to notice

⁸For each point $x \in \gamma_{\mathcal{O}}$, the set of spatial unit exterior vectors forms a one-dimensional hyperbola in Minkowski spacetime.

that the short distance $\epsilon > 0$, which plays the role of the cutoff for the EE, is indeed a geometrical quantity belonging to the continuum QFT.

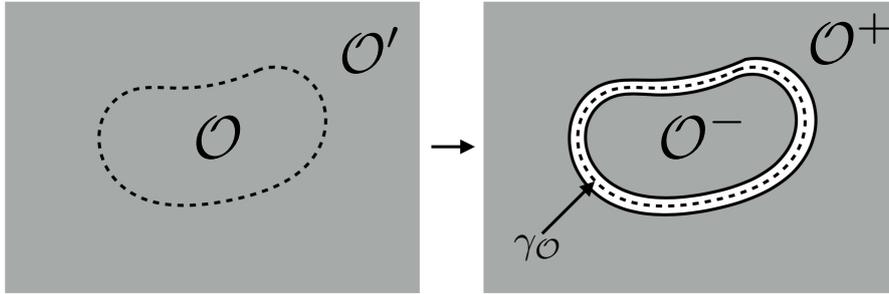


Figure 4.3: Geometrical setup for the definition of the EE through the MI.

S_{reg} is not a function of the region \mathcal{O} alone but depends on the framing, which includes the vector field η . As $\epsilon \rightarrow 0^+$ crossing all scales of the theory and curvature scales of $\gamma_{\mathcal{O}}$, we expect that S_{reg} can be written as a series of inverse integer powers in ϵ similar to (4.23). Since ϵ is a regulator, as we are taking the $\epsilon \rightarrow 0^+$ limit, we discard positive powers of ϵ . The divergent terms with negative exponents are produced by ultra-local entanglement between degrees of freedom arbitrarily close to both sides of $\gamma_{\mathcal{O}}$. In fact, since the cutoff ϵ is geometrical, we expect that the coefficients of these divergent contributions could be written as integrals of local geometrical quantities along $\gamma_{\mathcal{O}}$ like the one of (4.24). This geometric integrand is constructed with the metric, powers of the curvature of $\gamma_{\mathcal{O}}$, and the vector field η . The dependence on η should only show up in the divergent terms, and we should be able to subtract these terms to eliminate the frame dependence. Finally, we define

$$S(\mathcal{O}) := \lim_{\epsilon \rightarrow 0^+} [S_{reg}(\mathcal{O}; \epsilon, \eta) - \text{local divergent terms}] . \quad (4.45)$$

In consequence, $S(\mathcal{O})$ is finite, Poincaré invariant, and completely defined by the continuum theory itself. It can be thought of as a *minimally subtracted entanglement entropy*. We emphasize that it is free from ambiguities since it is uniquely and well-defined for any QFT.

While $S(\mathcal{O})$ does not have the property of being positive for arbitrary regions, it should retain some other important properties of entropy. The symmetry between complementary regions $S(\mathcal{O}) = S(\mathcal{O}')$ is built-in by definition and comes from the symmetry of the mutual information under the exchange of the two regions. $S(\mathcal{O})$ should satisfy strong subadditivity [32, 100]. This is important for example in the context of the entropic proof of the irreversibility theorems.

The extensivity and locality of the divergences in ϵ of $S_{reg}(\mathcal{O}; \epsilon, \eta)$ and the strong subadditivity property of the finite entropy $S(\mathcal{O})$ could be studied with complete mathematical rigor in AQFT, but this investigation has not been done yet.⁹

⁹See [106] for a recent related investigation.

4.3. Free fields in the lattice

One convenient way to regularize QFT is introducing a lattice cutoff. From the algebraic standpoint, this choice is the most natural, since the regularized QFT still behaves as a sort of AQFT. In this case, the degrees of freedom of QFT are described using a $d - 1$ dimensional square lattice \mathcal{L} . The square lattice \mathcal{L} has to be considered as a discretization, in cells of spacing δ , of the spacetime, over the Cauchy slice $\Sigma_0 := \{x \in \mathbb{R}^d : x^0 = 0\}$. In each point $j \in \mathcal{L}$, we place an operator algebra \mathfrak{A}_j , which could be a finite dimensional algebra (in the case of fermions) or a type I vN algebra (in the case of bosons). We define the global algebra as the infinite tensor product $\mathfrak{A} := \bigotimes_{j \in \mathcal{L}} \mathfrak{A}_j$. For any finite subset $A \subset \mathcal{L}$, we can define the subalgebra

$$V \subset \mathcal{L} \mapsto \mathfrak{A}_A := \bigotimes_{j \in A} \mathfrak{A}_j \subset \mathfrak{A}. \quad (4.46)$$

Such an algebra has to be considered as the regularization of the local algebra $\mathcal{A}(D(\mathcal{C}))$, where $D(\mathcal{C}) \subset \mathbb{R}^d$ is the Cauchy development of any open region $\mathcal{C} \subset \Sigma_0$, which contains all the lattice points $j \in A$. It is interesting to notice that the correspondence (4.46) satisfies the axioms 1, 2 and 3 (generating property, isotony and causality) of definition 2.56, but of course, it does not satisfy axiom 4 (Poincaré invariance). For any $A_1, A_2 \subset \mathcal{L}$ such that $A_1 \cap A_2 = \emptyset$, we have that $\mathfrak{A}_{A_1 \cup A_2} := \mathfrak{A}_{A_1} \otimes \mathfrak{A}_{A_2}$. Furthermore, we have that

$$\mathfrak{A} := \mathfrak{A}_A \otimes \mathfrak{A}_{A'}, \quad (4.47)$$

where $A' := \mathcal{L} - A$. In other words, the regularized algebras of complementary regions are statistically independent. In the cases of our interest, we also have that $\mathfrak{A}'_A = \mathfrak{A}_{A'}$, where \mathfrak{A}'_A is the commutant of \mathfrak{A}_A computed inside the global algebra $\mathfrak{A}_{\mathcal{L}}$. Given any state on the global algebra of QFT, it could be lifted to a global state ω in the regularized algebra \mathfrak{A} . In order to calculate the EE for the local algebra \mathfrak{A}_A , we need to compute the statistical operator ρ_A of the restricted state $\omega|_{\mathfrak{A}_A}$.

In QFT, the algebra \mathfrak{A} is usually described by a canonical commutation (resp. anti-commutation) relations algebra, usually denoted as CCR (resp. CAR) algebra. In many situations, we consider a Gaussian state ω on such an algebra. For example, the vacuum state in free field theories. The complete knowledge of a Gaussian state is entirely determined by the knowledge of its two-point correlators. In the following, we show how to compute the statistical operator ρ_A , and hence the modular Hamiltonian and the EE, for Gaussian states in CCR and CAR algebras.

Remark 4.9. Throughout the following subsections, \dagger denotes the Hermitian conjugate and $*$ denotes the complex conjugate.

4.3.1. CAR algebra (fermions)

We start with the CAR algebra of n degrees of freedom. Such an algebra is defined as the free $*$ -algebra generated by the operators ψ_1, \dots, ψ_n modulo the relations

$$\{\psi_j, \psi_k^\dagger\} = \delta_{jk} \quad \text{and} \quad \{\psi_j, \psi_k\} = \{\psi_j^\dagger, \psi_k^\dagger\} = 0. \quad (4.48)$$

It can be shown that the canonical representation of this algebra is the finite dimensional algebra $\mathfrak{A} := \bigotimes_{j=1}^n \mathfrak{A}_j$, where each subalgebra $\mathfrak{A}_j := M_2(\mathbb{C})$ represents a one fermion site.¹⁰ Therefore, in this case, we have a finite quantum system.

Given any state ω on \mathfrak{A} , the general structure of its two-points correlators is uniquely determined by¹¹

$$\langle \psi_j \psi_k^\dagger \rangle =: C_{jk} \quad \text{and} \quad \langle \psi_j \psi_k \rangle =: D_{jk}, \quad (4.51)$$

where $C, D \in \mathbb{C}^{n \times n}$. The other two-points correlators can be obtained from (4.51) imposing anticommutation relations and hermiticity of ω , namely

$$\langle \psi_j^\dagger \psi_k \rangle = \delta_{jk} - C_{kj} \quad \text{and} \quad \langle \psi_j^\dagger \psi_k^\dagger \rangle = D_{kj}^* = (D^\dagger)_{jk}. \quad (4.52)$$

Furthermore, the anticommutation relations impose $D = -D^T$, and the positivity of ω imposes $C > 0$ and $\mathbf{1}_n - C > 0$. This last fact implies that all the eigenvalues of C lie in the interval $[0, 1]$.

Let us introduce the following ‘‘charge’’ symmetry transformation. It is a continuous (global) symmetry given by a one-parameter group of automorphisms ϱ_s such that

$$\varrho_s(\psi_j) = e^{is} \psi_j \quad \text{and} \quad \varrho(\psi_j^\dagger) = e^{-is} \psi_j^\dagger. \quad (4.53)$$

From now on, we assume that the state is invariant under ϱ_s , i.e. $\omega = \omega \circ \varrho_s$.¹² This constrains (4.51) even more, imposing $D = 0$. For the structure of multipoint correlators, we now assume that the state ω is *Gaussian*, i.e. all odd-point correlators are zero and all even-point correlators are given by the usual Wick theorem [107]. Since $D = 0$, we also

¹⁰We denote by $(\sigma_j)_{j=1,2,3} \in M_2(\mathbb{C})$ the Pauli matrices. For one degree of freedom, the fermionic operators are represented as

$$\psi := \frac{\sigma_1 + i\sigma_2}{2} \in M_2(\mathbb{C}) \quad \text{and} \quad \psi^\dagger := \frac{\sigma_1 - i\sigma_2}{2} \in M_2(\mathbb{C}). \quad (4.49)$$

For n degrees of freedom, the fermionic operators are represented as

$$\psi_j^b := \Gamma \otimes \dots \otimes \Gamma \otimes \underset{(\text{site } j)}{\psi^b} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} \in M_2(\mathbb{C})^{\otimes n}, \quad b = \cdot, \dagger, \quad (4.50)$$

where $\Gamma := \sigma_3$.

¹¹Here we adopt the usual convention $\langle A \rangle := \omega(A)$.

¹²In QFT, this happens for any state of definite charge, and in particular, for the vacuum state.

have that all even-point correlators with distinct numbers of fermions and anti-fermions are zero. All non-zero multipoint correlators are obtained from the two-point functions according to

$$\left\langle \psi_{j_1} \cdots \psi_{j_l} \psi_{k_1}^\dagger \cdots \psi_{k_l}^\dagger \right\rangle = (-1)^{\frac{l(l-1)}{2}} \sum_{\sigma \in S_l} (-1)^{|\sigma|} \prod_{m=1}^l \left\langle \psi_{j_m} \psi_{k_{\sigma(m)}}^\dagger \right\rangle, \quad (4.54)$$

where S_l is the permutation group of l elements. It is a well-known fact that the state ω , and hence its statistical operator ρ , are completely determined by the multipoint correlators as above.

We now choose a proper subset of sites $A \subsetneq \mathcal{L} = \{1, \dots, n\}$ and define the bipartite system $\mathfrak{A} = \mathfrak{A}_A \otimes \mathfrak{A}_{A'}$ where

$$\mathfrak{A}_A := \bigotimes_{j \in A} \mathfrak{A}_j \quad \text{and} \quad \mathfrak{A}_{A'} := \bigotimes_{j \in A'} \mathfrak{A}_j. \quad (4.55)$$

To compute the entanglement entropy of the “region” A we need to first compute the statistical operator ρ_A of the restricted state $\omega|_{\mathfrak{A}_A}$. For that, we use the following ansatz

$$\rho_A := \frac{1}{Z} e^{-\sum_{j,k \in A} \psi_j^\dagger K_{jk} \psi_k}, \quad (4.56)$$

which always works for any Gaussian state [108]. $K \in \mathbb{C}^{n_A \times n_A}$, $n_A := \#A$, and $Z = \det(1 - e^{-K}) > 0$ in order to have $\text{Tr}_{\mathfrak{A}_A}(\rho_A) = 1$. Imposing the following relation

$$C_{jk} = \langle \psi_j \psi_k^\dagger \rangle = \frac{1}{Z} \text{Tr}_{\mathfrak{A}_A} \left(e^{-\sum_{j,k \in A} \psi_j^\dagger K_{jk} \psi_k} \psi_j \psi_k^\dagger \right), \quad (4.57)$$

a straightforward computation gives

$$K_A = -\log(C_A^{-1} - 1), \quad (4.58)$$

where C_A is just the matrix C_{jk} but where the indices are restricted to $j, k \in A$ [105]. In other words, if $\{|u_l\rangle\} \subset \mathbb{C}^{n_A}$ is a orthonormal basis of eigenvectors of C_A with the corresponding eigenvalues $\{\lambda_l\}$, then we have that

$$K_{jk} = -\sum_{l=1}^{n_A} \log(\lambda_l^{-1} - 1) |u_l\rangle_j \langle u_l|_k. \quad (4.59)$$

Since the eigenvalues of C_A are restricted to the interval $[0, 1]$, the above equation is valid whenever all $\lambda_l < 1$. When $\lambda_l = 1$ for some l , it means that the local reduced density matrix has a zero eigenvalue, which is equivalent to the reduced state not being faithful. In QFT, because of the Reeh-Schlieder theorem, this is forbidden for the vacuum state and

the algebra of any standard region \mathcal{O} .

When some eigenvalues are equal to 1, a more careful calculation shows that (4.59) still holds if we replace the sum including only eigenvalues less than 1. The fact that all eigenvalues are less than 1, means that the restricted state $\omega|_{\mathfrak{A}_A}$ is faithful.

The entanglement entropy and the entanglement Rényi entropies follow from (3.10) and (3.17)

$$S(A) = \sum_{l=1}^{n_A} \left[\log(1 - e^{-\lambda_l}) + \frac{\lambda_l}{1 + e^{\lambda_l}} \right] = -\text{Tr} [(1 - C_A) \log(1 - C_A) + C_A \log C_A] \quad (4.60)$$

$$S_\alpha(A) = \frac{1}{1 - \alpha} \sum_{l=1}^{n_A} [\log(1 - e^{-\alpha\lambda_l}) + \alpha \log(1 - e^{-\lambda_l})] = \frac{1}{1 - \alpha} \text{Tr} \log [(1 - C_A)^\alpha + C_A^\alpha],$$

where the above traces have to be taken in the finite dimensional algebra space $\mathbb{C}^{n_A \times n_A}$. The modular Hamiltonian is obtained from (4.56)

$$K_A = -\log(\rho_A) = \log(Z) \mathbf{1} + \sum_{j,k \in A} \psi_j^\dagger K_{jk} \psi_j. \quad (4.61)$$

4.3.2. CCR algebra (bosons)

We start with the CCR algebra of n degrees of freedom. Such an algebra is defined as the free $*$ -algebra generated by the operators $\phi_1, \dots, \phi_n, \pi_1, \dots, \pi_n$ modulo the relations

$$[\phi_j, \pi_k] = i\delta_{jk}, \quad [\phi_j, \phi_k] = [\pi_j, \pi_k] = 0, \quad (4.62)$$

$$\phi_j = \phi_j^\dagger \quad \text{and} \quad \pi_j = \pi_j^\dagger. \quad (4.63)$$

As we have explained in example 2.15, this algebra is not indeed a C^* -algebra. However, we could pass to the corresponding Weyl algebra (exponentiated version). We also have that the Weyl algebra is not finite dimensional. In fact, its unique irreducible representation is $\mathfrak{A} := \mathcal{B}(\mathcal{H})$ where $\mathcal{H} := L^2(\mathbb{R}^n)$. The operators act as

$$\phi_j \Psi(q_1, \dots, q_n) = q_j \Psi(q_1, \dots, q_n) \quad \text{and} \quad \pi_j \Psi(q_1, \dots, q_n) = -i \frac{\partial}{\partial q_j} \Psi(q_1, \dots, q_n) \quad (4.64)$$

The operators ϕ_j, π_k are unbounded, but we can use them to completely characterize a given state ω . Without loss of generality, we assume that $\langle \phi_j \rangle = \langle \pi_j \rangle = 0$ for all $j = 1, \dots, n$. The general structure of the two-point correlators of ω is uniquely determined by

$$\langle \phi_j \phi_k \rangle =: X_{jk}, \quad \langle \pi_j \pi_k \rangle =: P_{jk}, \quad (4.65)$$

$$\langle \phi_j \pi_k \rangle =: \frac{i}{2} \delta_{jk} + D_{jk}, \quad (4.66)$$

where $X, P, D \in \mathbb{C}^{n \times n}$. We also have the following constraints: $X, P \in \mathbb{R}^{n \times n}$ due to the hermiticity of the state, $X, P > 0$ due to the positivity of the state and $D \in \mathbb{R}^{n \times n}$ due to the commutation relations. The positivity of $\langle (\phi_j + ic_{jk}\pi_k)^\dagger (\phi_j + ic_{jk}\pi_k) \rangle$ for arbitrary numbers $c_{jk} \in \mathbb{R}$ implies that $XP > \frac{1}{4}$, which means that all the eigenvalues of XP lie in the interval $[\frac{1}{4}, +\infty)$.

Let us now introduce the ‘‘time inversion’’ symmetry. It is given by an anti-automorphism τ such that

$$\tau(\phi_j) = \phi_j \quad \text{and} \quad \tau(\pi_j) = -\pi_j. \quad (4.67)$$

We now assume that the state is invariant under τ , i.e. $\omega = \omega \circ \tau$. This constrains (4.51) more, imposing $D = 0$. Like in the fermion case, we assume that the state ω is Gaussian. Then, all odd-point correlators are zero and all even-point correlators are given by the usual Wick theorem

$$\langle f_{j_1} \cdots f_{j_{2l}} \rangle = \frac{1}{2^l l!} \sum_{\sigma \in S_{2l}} \prod_{m=1}^l \langle f_{j_{2m-1}} f_{j_{\sigma(2m)}} \rangle, \quad (4.68)$$

where f_j are any of the field or momentum variables, and the operators are ordered inside the expectation value so that all the field variables are placed on the left and all the momentum variables on the right. It can be shown that the state ω can be uniquely determined by the collection of all multipoint correlators.

Once we have characterized the structure of the state, we compute the entanglement entropy. For such a purpose, it is useful to notice the following decomposition of the algebra \mathfrak{A} ,

$$\mathfrak{A} := \bigotimes_{j=1}^n \mathfrak{A}_j, \quad \text{where} \quad \mathfrak{A}_j := \mathcal{B}(L^2(\mathbb{R})). \quad (4.69)$$

As in the fermion case, we choose a proper subset of sites $A \subsetneq \mathcal{L} = \{1, \dots, n\}$ and define the bipartite system $\mathfrak{A} = \mathfrak{A}_A \otimes \mathfrak{A}_{A'}$, according to (4.55). Even though the algebras are infinite dimensional, we can still use all the formulas for finite quantum systems to compute any information measure.¹³ In particular, the statistical operator ρ_A of the restricted state $\omega|_{\mathfrak{A}_A}$ is a well-defined density matrix.¹⁴ We use the following ansatz

$$\rho_A := \frac{1}{Z} e^{-\sum_{j,k \in A} (\pi_j N_{jk} \pi_k + \phi_j M_{jk} \phi_k)}, \quad (4.70)$$

which always works for any Gaussian state [109, 110]. Similarly, as we did in the fermion

¹³This is because, as we discuss at the beginning of section 3.3, these algebras are type I vN algebras.

¹⁴If ω is a normal state (described by a density matrix in \mathcal{H}) then $\omega|_{\mathfrak{A}_V}$ must be normal. In fact, ρ_V is given by the usual partial trace on $\mathfrak{A}_{V'}$.

case, we impose (4.65) to the above ansatz. A straightforward calculation gives

$$M = P \frac{1}{2C} \log \left(\frac{C + \frac{1}{2}}{C - \frac{1}{2}} \right), \quad (4.71)$$

$$N = \frac{1}{2C} \log \left(\frac{C + \frac{1}{2}}{C - \frac{1}{2}} \right) X, \quad (4.72)$$

where $C := \sqrt{XP}$ [105]. We remark that the matrices X, P, C in the above relations are the restriction to A of the original $n \times n$ matrices, i.e. they are square matrices of dimension $n_A := \#A$. Expressions (4.71) and (4.72) can be computed using the eigenvectors and eigenvalues of XP and PX .

The entanglement entropy and the entanglement Rényi entropies follow from (3.10) and (3.17)

$$S(A) = \text{Tr} \left[\left(C + \frac{1}{2} \right) \log \left(C + \frac{1}{2} \right) - \left(C - \frac{1}{2} \right) \log \left(C - \frac{1}{2} \right) \right], \quad (4.73)$$

$$S_\alpha(A) = \frac{1}{\alpha - 1} \text{Tr} \log \left[\left(C + \frac{1}{2} \right)^\alpha + \left(C - \frac{1}{2} \right)^\alpha \right]. \quad (4.74)$$

Finally, the modular Hamiltonian is

$$K_A = -\log(\rho_A) = \log(Z) \mathbf{1} + \sum_{j,k \in A} (\pi_j N_{jk} \pi_k + \phi_j M_{jk} \phi_k). \quad (4.75)$$

4.3.3. General CCR algebra

Here we generalize the previous result to a general CCR algebra. A general CCR algebra is defined as the free $*$ -algebra generated by the operators f_1, \dots, f_{2n} modulo the relations

$$[f_j, f_k] =: iC_{jk} \quad \text{and} \quad f_j = f_j^\dagger, \quad (4.76)$$

where $C \in \mathbb{R}^{n \times n}$ and $C = -C^T$ [111]. It is useful to define the operator vector $\bar{f} := (f_1, \dots, f_{2n})^T$.

A general Gaussian state ω has a two-point correlator

$$\langle f_j f_k \rangle =: F_{jk}, \quad (4.77)$$

with $F \in \mathbb{C}^{n \times n}$, and $F > 0$ due to the positivity of the state. Due to the commutation relations it must be that

$$C_{jk} = 2 \text{Im}(F_{jk}). \quad (4.78)$$

Now, we show how to map this problem to the previous one in section 4.3.2. Since C is skew-symmetric and real, there exist $O \in \mathbb{R}^{2n \times 2n}$, $O^{-1} = O^T$ and $D \in \mathbb{R}^{n \times n}$, $D > 0$ and

diagonal, such that

$$OCO^T = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}. \quad (4.79)$$

Defining

$$Q = \begin{pmatrix} D^{-\frac{1}{2}} & 0 \\ 0 & D^{-\frac{1}{2}} \end{pmatrix}, \quad (4.80)$$

we have that $Q \in \mathbb{R}^{2n \times 2n}$, $Q > 0$, and we can write the original commutator in the canonical form, i.e.

$$QOCO^TQ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.81)$$

Accordingly,

$$\bar{\Phi} = (\phi_1, \dots, \phi_n, \pi_1, \dots, \pi_n)^T := QO\bar{f}, \quad (4.82)$$

are $2n$ variables, satisfying the CCR algebra of the previous section. Implementing all the same changes to the state (4.77), we can write

$$QOFO^TQ =: \begin{pmatrix} X & \frac{i}{2} + D \\ -\frac{i}{2} + D^\dagger & P \end{pmatrix}, \quad (4.83)$$

where $X, P, D \in \mathbb{R}^{n \times n}$ and $X, P > 0$ due to (4.78) and the positivity of the state. Comparing this last expression with (4.65) and (4.66), we conclude that the matrix (4.83) has the information about the correlations of the state ω in the new variables (4.82). As in the previous case, we assume that $D = 0$. This is a consequence of the time-inversion invariance of the state respect to the anti-automorphism (4.67). In term of the variables \bar{f} , there is an anti-automorphism τ such that $\tau(f_i) = \sum_{j=1}^{2n} T_{ij}f_j$ with $T \in \mathbb{R}^{2n \times 2n}$ and $TCT^\dagger = -C$, and such the state ω is invariant, i.e $\omega \circ \tau = \omega$.

Now we choose a proper even subset of sites $A \subsetneq \mathcal{L} = \{1, \dots, 2n\}$ and define $n_A := \frac{\#A}{2}$. In order to get well-defined subsystems, we must assume that the matrix C_{jk} restricted to the indices $j, k \in A$ is also skew-symmetric. In other words, the subalgebra generated by $\{f_j\}_{j \in A}$ forms itself a general CCR algebra. In terms of the new variables $\bar{\Phi}$, this means that a given field operator ϕ_j belongs to the subsystem if and only if its corresponding canonical conjugate π_j belongs to the subsystem. Now, we are able to consider the above procedure but for the subalgebra \mathfrak{A}_A , which is defined by the variables $\{f_j\}_{j \in A}$. Then, we have that $X, P, D \in \mathbb{R}^{n_A \times n_A}$ and $Q, O \in \mathbb{R}^{2n_A \times 2n_A}$.

The statistical operator ρ_A of the restricted state $\omega|_{\mathfrak{A}_A}$ could be expressed in terms of the variables $\bar{\Phi}$ as in (4.70), where the matrix kernels M, N are given by the expressions (4.71) and (4.72). In this case, the kernel $C = \sqrt{XP}$ has to be read from the diagonal blocks of (4.83). However, it is useful to have an equivalent expression in terms of the

original variables \bar{f} . For that, we define the matrix $V \in \mathbb{C}^{2n_A \times 2n_A}$

$$V := -iC^{-1}F - \frac{1}{2} = O^T Q \begin{pmatrix} \frac{1}{2} & iP \\ -iX & \frac{1}{2} \end{pmatrix} Q^{-1} O, \quad (4.84)$$

whose spectrum is of the form $\{\pm\lambda_j : \lambda_j \geq \frac{1}{2} \text{ for } j = 1, \dots, n_A\}$ [39]. A straightforward computation reveals [39]

$$\rho_A := \frac{1}{Z} e^{-\sum_{j,k \in A} f_j K_{jk} f_k}, \quad (4.85)$$

where the matrix $K \in \mathbb{C}^{2n_A \times 2n_A}$ is given by

$$K := -\frac{i}{2} \frac{V}{|V|} \log \left(\frac{|V| + \frac{1}{2}}{|V| - \frac{1}{2}} \right) C^{-1}. \quad (4.86)$$

Then, the entanglement entropy and the Rényi entanglement entropy are [39]

$$\begin{aligned} S(A) &= \text{Tr} \left[\left(V + \frac{1}{2} \right) \log \left| V + \frac{1}{2} \right| \right] \\ &= \text{Tr} \Theta(V) \left[\left(V + \frac{1}{2} \right) \log \left(V + \frac{1}{2} \right) - \left(V - \frac{1}{2} \right) \log \left(V - \frac{1}{2} \right) \right], \end{aligned} \quad (4.87)$$

$$S_\alpha(A) = \frac{1}{\alpha - 1} \text{Tr} \Theta(V) \log \left[\left(V + \frac{1}{2} \right)^\alpha + \left(V - \frac{1}{2} \right)^\alpha \right], \quad (4.88)$$

where $\Theta(V)$ is the orthogonal projector on the subspace of positive eigenvalues of V . The modular Hamiltonian is

$$K_A = -\log(\rho_A) = \log(Z) \mathbf{1} + \sum_{j,k \in A} f_j K_{jk} f_k. \quad (4.89)$$

4.3.4. Continuum limit

Here we make a few comments on how the above expressions have to be understood in the continuum limit. Continuum limit has to be thought of as taking the limit of vanishing lattice spacing, i.e. $\delta \rightarrow 0^+$. Such a limit gives place to a divergent EE. However, it gives meaningful expressions for the correlators kernels and the modular Hamiltonian. Here we show this for the case of the CAR algebra. In the lattice model, the modular Hamiltonian is given by the expression (4.61). It is important to emphasize that the term proportional to the identity operator is spurious for the purpose of the modular flow,

$$\sigma_t(O) = e^{iK_A t} O e^{-iK_A t} = e^{it \sum_{j,k \in A} \psi_j^\dagger K_{jk} \psi_j} O e^{-it \sum_{j,k \in A} \psi_j^\dagger K_{jk} \psi_j}, \quad O \in \mathfrak{A}_V. \quad (4.90)$$

Furthermore, in QFT such a constant term is indeed infinite. Such an infinity comes from the fact that the (4.61) is an expression for the “inner” modular Hamiltonian, which, in

the continuum QFT, is a sesquilinear form rather than an operator. For this reason, in QFT we denote the “inner” modular Hamiltonian just by

$$K_A = \sum_{j,k \in A} \psi_j^\dagger K_{jk} \psi_j. \quad (4.91)$$

The translation of the above equations to the continuum QFT is as follows. First we pick a spatial region $\mathcal{C} \subset \Sigma_0 := \{x \in \mathbb{R}^d : x^0 = 0\}$. The vacuum state ω_0 gives place to the two-point correlator kernel $\omega_0(\psi(\bar{x})\psi^\dagger(\bar{y})) = \langle \psi(\bar{x})\psi^\dagger(\bar{y}) \rangle =: C(\bar{x} - \bar{y})$ where $\bar{x}, \bar{y} \in \mathcal{C}$. This correlator can be understood as a Hermitian kernel operator acting on the “one-particle” Hilbert space $L^2(\mathcal{C}) \otimes \mathbb{C}^S$ through the relation¹⁵

$$(Cf)(\bar{x}) = \int_{\mathcal{C}} C(\bar{x} - \bar{y}) f(\bar{y}) d^{d-1}y. \quad (4.92)$$

Then, the relation (4.58) must be understood as an operator equation, where the action of the operators is defined through their kernels. In fact, the Hermitian correlator kernel has a complete set of eigenfunctions $|u_{s,k}\rangle = u_{s,k}(\bar{x})$ satisfying¹⁶

$$C|u_{s,k}\rangle = \frac{1 + \tanh(\pi s)}{2} |u_{s,k}\rangle, \quad (4.93)$$

$$\int_{\mathcal{C}} u_{s,k}(\bar{x})^\dagger u_{s',k'}(\bar{x}) d^{d-1}x = \delta_{k,k'} \delta(s - s'), \quad (4.94)$$

$$\sum_{k \in \Upsilon} \int_{\mathbb{R}} u_{s,k}(\bar{x}) u_{s,k}(\bar{y})^\dagger ds = \delta(\bar{x} - \bar{y}), \quad (4.95)$$

where $s, s' \in \mathbb{R}$, $\bar{x}, \bar{y} \in \mathcal{C}$ and $k \in \Upsilon$ is a possible degeneracy index.¹⁷ Of course, the above eigenfunctions depend strongly on the choice of the region \mathcal{C} . Using such eigenfunctions, the correlator kernel could be expressed as

$$C(\bar{x} - \bar{y}) = \sum_{k \in \Upsilon} \int_{\mathbb{R}} u_{s,k}(\bar{x}) \frac{1 + \tanh(\pi s)}{2} u_{s,k}(\bar{y})^\dagger ds. \quad (4.96)$$

Using expressions (4.58) and (4.91), we also have that

$$K = \int_{\mathcal{C} \times \mathcal{C}} d^{d-1}x d^{d-1}y \psi^\dagger(\bar{x}) \mathcal{K}(\bar{x}, \bar{y}) \psi(\bar{y}), \quad (4.97)$$

$$\mathcal{K}(\bar{x}, \bar{y}) = \sum_{k \in \Upsilon} \int_{\mathbb{R}} u_{s,k}(\bar{x}) 2\pi s u_{s,k}(\bar{y})^\dagger ds. \quad (4.98)$$

¹⁵The space \mathbb{C}^S appears because of the spin character of the field operator $\psi(\bar{x})$.

¹⁶In chapter 6 we show this explicitly for the chiral fermion field.

¹⁷Eventually, $k \in \Upsilon$ could be a continuous index, and hence, the summation and Kronecker delta function above have to be replaced by an integral and a Dirac delta function.

It is important to remark that the modular Hamiltonian (4.97) has to be understood in the sense of modular evolution according to formula (4.90). In fact, for any operator O belonging to the algebra $\mathcal{A}(D(\mathcal{C}))$, the modular evolution (4.90) is given by the usual Heisenberg equation

$$\partial_t O(t) = i[K, O(t)], \quad (4.99)$$

where $O(t) := \sigma_t(O)$. In particular, for an operator of the form $O_f := \int_{\mathcal{C}} \psi^\dagger(\bar{x}) f(\bar{x}) d^{d-1}x$ with $\text{sup}(f) \subset \mathcal{C}$, equation (4.99) implies

$$O_f(t) = \int_{\mathcal{C}} \psi^\dagger(\bar{x}) f(\bar{x}; t) d^{d-1}x, \quad (4.100)$$

$$\partial_t f(\bar{x}; t) = i \int_{\mathcal{C}} \mathcal{K}(\bar{x}, \bar{y}) f(\bar{y}; t) d^{d-1}y. \quad (4.101)$$

We define the operators

$$\tilde{\psi}_{s,k}^\dagger := \int_{\mathcal{C}} \psi^\dagger(\bar{x}) u_{s,k}(\bar{x}) d^{d-1}x \quad \text{and} \quad \tilde{\psi}_{s,k} := \int_{\mathcal{C}} u_{s,k}(\bar{x})^\dagger \psi(\bar{x}) d^{d-1}x, \quad (4.102)$$

which satisfy the CAR relations

$$\left\{ \tilde{\psi}_{s,k}, \tilde{\psi}_{s',k'}^\dagger \right\} = \delta_{k,k'} \delta(s - s'), \quad \left\{ \tilde{\psi}_{s,k}, \tilde{\psi}_{s',k'} \right\} = \left\{ \tilde{\psi}_{s,k}^\dagger, \tilde{\psi}_{s',k'}^\dagger \right\} = 0. \quad (4.103)$$

Modes (4.102) diagonalize the modular Hamiltonian in the sense that they evolve under the modular flow accordingly to

$$\sigma_t \left(\tilde{\psi}_{s,k}^\dagger \right) = e^{i2\pi st} \tilde{\psi}_{s,k}^\dagger \quad \text{and} \quad \sigma_t \left(\tilde{\psi}_{s,k} \right) = e^{-i2\pi st} \tilde{\psi}_{s,k}. \quad (4.104)$$

Moreover, it is not difficult to show that the formulas (4.104) imply the KMS-condition for the two-point correlator

$$\left\langle \sigma_{t-i} \left(\tilde{\psi}_{s,k} \right) \tilde{\psi}_{s',k'}^\dagger \right\rangle = \left\langle \tilde{\psi}_{s',k'}^\dagger \sigma_t \left(\tilde{\psi}_{s,k} \right) \right\rangle, \quad \forall \bar{x}, \bar{y} \in \mathcal{C}, \forall s, s' \in \mathbb{R}. \quad (4.105)$$

Since the set of operators $\{\psi_{s,k}(\bar{x}), \psi_{s,k}^\dagger(\bar{x}) : s \in \mathbb{R}, k \in \Upsilon\}$ form a “complete basis” of the algebra of operators of the region \mathcal{C} and the multipoint correlators are determined by the two-point correlators, we have that equations (4.102) and (4.104) define rigorously the modular flow in the continuum QFT. Of course, the information about the state and the region (algebra) in such equations is completely codified in the eigenfunctions $u_{s,k}(\bar{x})$.

The same analysis could be carried out for the EE. According to (4.60), in the continuum

QFT, we shall have

$$S(\mathcal{C}) = \text{Tr}(\mathcal{S}(\bar{x}, \bar{y})) = \sum_{k \in \Upsilon} \int_{\mathbb{R}} ds \int_{\mathcal{C} \times \mathcal{C}} d^{d-1}x d^{d-1}y u_{s,k}(\bar{x})^\dagger \mathcal{S}(\bar{x}, \bar{y}) u_{s,k}(\bar{y}), \quad (4.106)$$

where

$$\mathcal{S}(\bar{x}, \bar{y}) = \sum_{k \in \Upsilon} \int_{\mathbb{R}} ds u_{s,k}(\bar{x}) g(s) u_{s,k}(\bar{y})^\dagger, \quad (4.107)$$

$$g(s) = \pi s \tanh(\pi s) + \log(\text{sech}(\pi s)) - \log(2), \quad (4.108)$$

is the spectral resolution of $(-1)[(1-C)\log(1-C) + C\log C]$ according to (4.96). Performing the integral over $d^{d-1}y$ in equation (4.106) we get

$$S(\mathcal{C}) = \sum_{k \in \Upsilon} \int_{\mathbb{R}} ds g(s) \int_{\mathcal{C}} d^{d-1}x u_{s,k}(\bar{x})^\dagger u_{s,k}(\bar{x}). \quad (4.109)$$

If we perform the integral over $d^{d-1}x$ we obtain a $\delta(0)$ factor, giving a divergent EE. This happens because, when we have moved to the continuum QFT, we have lost the original lattice cutoff δ . In order to regularize the entropy (4.109), we introduce a new geometrical cutoff which is given by changing the integration region \mathcal{C} by the “regularized” region $\mathcal{C}_\epsilon \subset \mathcal{C}$ defined to contain all the points of \mathcal{C} that are separated by a distance greater than ϵ from the boundary $\partial\mathcal{C}$, i.e. $\mathcal{C}_\epsilon := \{\bar{x} \in \mathcal{C} : \text{dist}(\bar{x}, \partial\mathcal{C}) > \epsilon\}$. For example, for a $d = 2$ QFT and a finite interval $\mathcal{C} := (a, b)$, we have that $\mathcal{C}_\epsilon = (a + \epsilon, b - \epsilon)$. Then, we have defined the following “geometrically” regularized (and finite) EE

$$S_\epsilon(\mathcal{C}) = \sum_{k \in \Upsilon} \int_{\mathbb{R}} ds g(s) \int_{\mathcal{C}_\epsilon} d^{d-1}x u_{s,k}(\bar{x})^\dagger u_{s,k}(\bar{x}). \quad (4.110)$$

The interesting fact arises when we consider the MI for two non-intersecting regions $\mathcal{C}_1, \mathcal{C}_2 \subset \Sigma_0$. In this case, we have that

$$S_\epsilon(\mathcal{C}_1) + S_\epsilon(\mathcal{C}_2) - S_\epsilon(\mathcal{C}_1 \cup \mathcal{C}_2) \xrightarrow{\epsilon \rightarrow 0^+} I(\mathcal{C}_1, \mathcal{C}_2). \quad (4.111)$$

This is because the divergences of each of the terms are local and extensive along the boundaries of the regions and they cancel out. In fact, using the kernel (4.107) of the regions $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_1 \cup \mathcal{C}_2$, we can define the kernel

$$\mathcal{I}_{\mathcal{C}_1, \mathcal{C}_2}(\bar{x}, \bar{y}) := \mathcal{S}_{\mathcal{C}_1}(\bar{x}, \bar{y}) \Theta_{\mathcal{C}_1}(\bar{y}) + \mathcal{S}_{\mathcal{C}_2}(\bar{x}, \bar{y}) \Theta_{\mathcal{C}_2}(\bar{y}) - \mathcal{S}_{\mathcal{C}_1 \cup \mathcal{C}_2}(\bar{x}, \bar{y}), \quad (4.112)$$

where $\Theta_{\mathcal{C}_j}$ is the characteristic function of the region \mathcal{C}_j . Equation (4.111) says that (4.112)

is a trace-class operator on the Hilbert space $\mathfrak{H} = L^2(\mathcal{C}_1 \cup \mathcal{C}_2) \otimes \mathbb{C}^S$ and that

$$\mathrm{Tr}_{\mathfrak{H}}(\mathcal{I}_{\mathcal{C}_1, \mathcal{C}_2}) = I(\mathcal{C}_1, \mathcal{C}_2) . \quad (4.113)$$

To summarize, we have shown how lattice formulas (for free fields) are still valid in the continuum QFT whenever they are interpreted appropriately. In fact, the eigenfunctions of the correlator kernel $C(\bar{x} - \bar{y})$ restricted to the region \mathcal{C} , contain all the information of the modular flow and the mutual information for two non-intersecting spatial regions.

Chapter 5

Relative entropy for coherent states

In this chapter, we make a rigorous computation of the RE between the vacuum state and a coherent state for a free scalar in the framework of AQFT. We study the case of the Rindler Wedge. Previous calculations, including path integral methods and computations from the lattice, give a result for this RE involving integrals of expectation values of the energy-momentum stress tensor along the considered region. However, the stress tensor is in general non-unique. That means that, if we start with some stress tensor, we can “improve” it by adding a conserved term without modifying the Poincaré charges. On the other hand, the presence of such an improving term affects the naive result for the RE by adding a non-vanishing boundary contribution along the entanglement surface. In other words, there is an ambiguity in the usual formula for the RE coming from the non-uniqueness of the stress tensor. The main motivation of this section is to solve this puzzle. We first show that all choices of stress tensor, except the canonical one, are not allowed by positivity and monotonicity of the RE. Then, we fully compute the RE between the vacuum and a coherent state in the framework of AQFT using the Araki formula and the techniques of modular theory explained in section 3.3. Both results coincide and give the usual expression for the RE calculated with the canonical stress tensor.

The chapter is organized as follows. First, in section 5.1, we show how ambiguities can arise when one computes the RE making naive calculations. We show, in the case of the Rindler wedge, that such ambiguities are related to the choice of an improvement term in the stress-tensor. Next, in section 5.2, we explain how to fix such ambiguities imposing bounds which come from the positivity and monotonicity of the RE. In the rest of the chapter, we compute the RE using Araki formula. In section 5.3, we briefly review the algebraic formulation of the free real scalar field. Because of a forthcoming necessity, we consider two different approaches. The first one is the usual approach, where we define a net of algebras associated with spacetime regions. The second one consists in defining the local algebras associated with spatial sets belonging to a common Cauchy surface. We also explain how both approaches are related. The reader who is familiar with these

concepts may skip this section and go directly to 5.4. There we study general aspects concerning the RE for coherent states which apply to any region, and next, in section 5.5, we explicitly compute the proposed RE for the Rindler wedge algebra. We study separately, the (trivial) case when the coherent state is made of a unitary operator belonging to the wedge algebra, and the more interesting (and also more difficult) case, when the coherent state has a non-vanishing density along the entanglement surface. We provide a complete mathematical rigorous proof of all the results. The proof of some theorems and some tedious, but straightforward calculations, are placed into appendices.

This chapter is based on the article [42].

5.1. Ambiguities in the heuristic formula for relative entropy

Here we show how ambiguities can arise when we try to compute entanglement quantities making naive calculations. To be concrete, we start with a QFT in the vacuum representation, and we want to compute the relative entropy $S_{\mathcal{W}_R}(\phi | \omega_0)$ between the vacuum state ω_0 and any other global pure state ϕ for the right Rindler wedge region \mathcal{W}_R . In actual computations, it is customary and useful to assume a cutoff model, such as a lattice, and proceed to the computation taking the continuum limit as a final step. In general, we expect the quantity computed belongs to the continuum theory as far as the result does not depend on the regularization. In the presence of a regularization, we can assume that the local algebras $\mathcal{A}(\mathcal{W}_R)$ and $\mathcal{A}(\mathcal{W}'_R)$ are statistically independent

$$\mathcal{B}(\mathcal{H}_0) \cong \mathcal{A}(\mathcal{W}'_R) \otimes \mathcal{A}(\mathcal{W}_R) , \quad (5.1)$$

and furthermore they are type I vN algebras. Then, we are able to use all the formulas derived in section 3.2. In particular, if ρ_0^R and ρ_ϕ^R are the statistical operators of the restricted states $\omega_0|_{\mathcal{A}(\mathcal{W}_R)}$ and $\phi|_{\mathcal{A}(\mathcal{W}_R)}$, then we can use the expressions (3.24) and (3.25), namely

$$S_{\mathcal{W}_R}(\phi | \omega_0) = \Delta \langle K_0^R \rangle - \Delta S_{\mathcal{W}_R} , \quad (5.2)$$

where

$$\Delta \langle K_0^R \rangle = \phi(K_0^R) - \omega_0(K_0^R) , \quad (5.3)$$

$$\Delta S_{\mathcal{W}_R} = S_{\mathcal{W}_R}(\phi) - S_{\mathcal{W}_R}(\omega_0) , \quad (5.4)$$

and $K_0^R = -\log(\rho_0^R)$ is the “inner” modular Hamiltonian of the vacuum vector $|0\rangle$.

One has to be very careful when using expressions like (5.2) in QFT. The l.h.s is a

well-defined quantity in the continuum QFT, but a mathematically rigorous definition of the continuum limit of the two terms on the r.h.s. of (5.2) has not been worked out in the literature yet. In fact, the EE entropy and the “inner” modular Hamiltonian K_0 are not well-defined quantities of the continuum QFT. As we have explained in section 4.1.1, the “inner” modular Hamiltonian K_0^R could be obtained cutting the “full” modular Hamiltonian K_0 into two pieces

$$K_0 = K_0^R - K_0^L, \quad (5.5)$$

where K_0^R belongs to $\mathcal{A}(\mathcal{W}_R)$ and K_0^L belongs to $\mathcal{A}(\mathcal{W}_L)$. In the case of the right Rindler wedge, it is given by (4.9)

$$K_0^R = 2\pi \int_{x^1 > 0} d^{d-1}x x^1 T_{00}(x). \quad (5.6)$$

This is not a well-defined operator in the Hilbert space because its fluctuation $\langle 0 | (K_0^R)^2 | 0 \rangle$ diverges. However, (5.6) is a well-defined sesquilinear form, and hence, expectation values as $\Delta \langle K_0^R \rangle$ can still be computed. Another more important issue is that cutting the “full” modular Hamiltonian in two pieces generates ambiguities. We are allowed for example to add field operators localized at the boundary such that K_0^R has still the same localization and commutation relations with operators inside \mathcal{W}_R . Another view of the same problem is that, hidden in expression (5.6), there is an ambiguity related to the non-uniqueness of the stress tensor. For example, for the free real scalar field, starting from the canonical stress tensor

$$T_{\mu\nu}^{can} =: \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \eta_{\mu\nu} (\partial_\sigma \varphi \partial^\sigma \varphi - m^2 \varphi^2) :, \quad (5.7)$$

we can add an “improving term” to obtain a new stress tensor

$$T_{\mu\nu} = T_{\mu\nu}^{can} + \frac{\lambda}{2\pi} (\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2) : \varphi^2 :, \quad (5.8)$$

where λ could be, in principle, any real number. The Poincaré generators obtained from (5.8) equal the ones obtained from (5.7), since both expressions differ in a boundary term which vanishes when the integration region is the whole space. However, the expression (5.6) for K_0^R involves an integral in a semi-infinite region, and hence, the presence of an improving term adds a non zero extra boundary term to the result,

$$K_R \rightarrow K_R + \lambda \int_{x^1=0} d^{d-2}x : \varphi^2(x) :. \quad (5.9)$$

This is essentially the only boundary term we can add with the correct dimensions, and that does not require a dimensionful coefficient with negative dimensions. This can have non-zero expectation values for certain states and makes the definition of $\Delta \langle K_0^R \rangle$ ambiguous.

Since the relative entropy is well-defined, this ambiguity must be compensated by an-

other one in the definition of $\Delta S_{\mathcal{W}_R}$ in (5.4). This is the subtraction of two divergent quantities, and again, we do not have a mathematically rigorous definition in the continuum QFT. We can make this definition unambiguous in a natural way by using the universal regularization of EE explained in section 4.2.3. Using such a universal prescription, one can define

$$\Delta S_{\mathcal{W}_R} = \lim_{\epsilon \rightarrow 0^+} [S_{reg}^\phi(\mathcal{W}_R; \epsilon, \eta^1) - S_{reg}^{\omega_0}(\mathcal{W}_R; \epsilon, \eta^1)] , \quad (5.10)$$

where we have chosen the unit exterior vector $\eta^1 := (0, -1, \bar{0})$. In fact, according to equation (4.42), each term of (5.10) is a MI between the regions $\mathcal{W}_{R,\epsilon} := \{x \in \mathbb{R}^d : x^1 > |x^0| + \frac{\epsilon}{2}\}$ and $\mathcal{W}_{L,\epsilon} := \mathcal{W}'_{R,-\epsilon}$ (see figure 5.1). One expects that the local ambiguities due to the cutoff in both terms of (5.10) cancel out, giving a finite $\Delta S_{\mathcal{W}_R}$. Defining $\Delta S_{\mathcal{W}_R}$ rigorously through (5.10), then $\Delta \langle K_0^R \rangle$ is also unambiguously defined through

$$\Delta \langle K_0^R \rangle = S_{\mathcal{W}_R}(\phi | \omega_0) + \Delta S_{\mathcal{W}_R} . \quad (5.11)$$

Then, the question that arises is whether this definition agrees with the expectation value of (5.6). In such a case, boundary terms in this expression should be automatically fixed. In particular, we should be able to study which value of the improvement term is the correct one for the free real scalar field in (5.8).

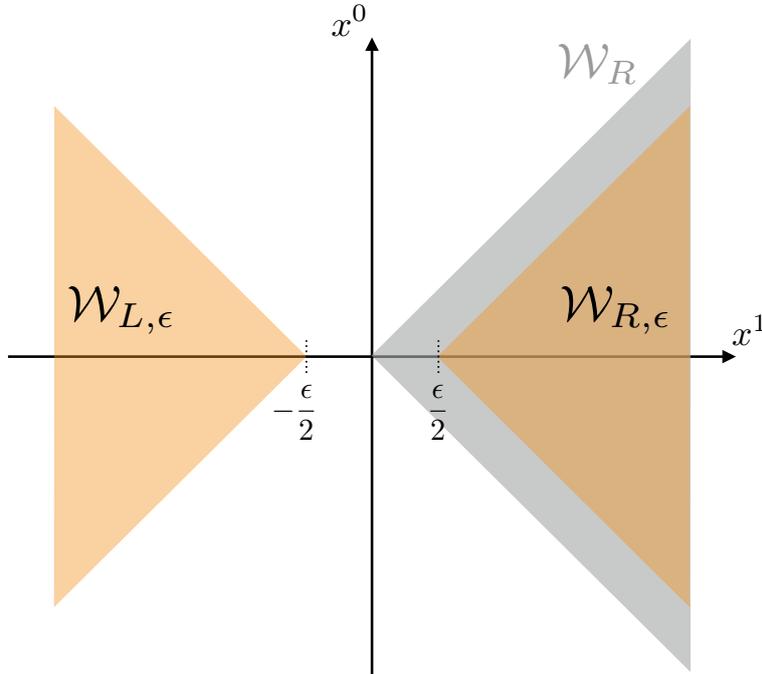


Figure 5.1: Regularized wedge regions for the computation of $\Delta S_{\mathcal{W}_R}$ according to formula (5.10).

In order to settle this issue, we analyze the relative entropy between a coherent state and the vacuum for a free real scalar field in the Rindler wedge. A coherent state ϕ is a global pure state formed out by acting on the vacuum vector with a unitary operator that

is the exponential of the smeared field, i.e.

$$\phi(\cdot) = \langle \Phi | \cdot | \Phi \rangle \quad \text{with} \quad |\Phi\rangle = e^{i \int d^{d-1}x [\varphi(\bar{x})f_\varphi(\bar{x}) + \pi(\bar{x})f_\pi(\bar{x})]} |0\rangle, \quad (5.12)$$

where $\varphi(\bar{x}) := \varphi(0, \bar{x})$, $\pi(\bar{x}) := \partial_0 \varphi(0, \bar{x})$, and $f_\varphi, f_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$.¹ For the purpose of the definition (5.10), we can represent the same state ϕ with a different vector $|\tilde{\Phi}\rangle = U_L U_R |0\rangle$ where U_R, U_L are unitaries, $U_R \in \mathcal{A}(\mathcal{W}_R)$ and $U_L \in \mathcal{A}(\mathcal{W}_L)$. Indeed, we can replace each of the smooth functions $f_\varphi(\bar{x})$ and $f_\pi(\bar{x})$ in (5.12) by the sum of two new smooth functions

$$f_\varphi \mapsto f_{\varphi,R} + f_{\varphi,L}, \quad f_\pi \mapsto f_{\pi,R} + f_{\pi,L}, \quad (5.13)$$

such that $f_{\varphi,R}(\bar{x}) = 0$ for $x^1 \leq 0$, $f_{\varphi,R}(\bar{x}) = f_\varphi(\bar{x})$ for $x^1 \geq \frac{\epsilon}{2}$, $f_{\varphi,L}(\bar{x}) = 0$ for $x^1 \geq 0$ and $f_{\varphi,L}(\bar{x}) = f_\varphi(\bar{x})$ for $x^1 \leq -\frac{\epsilon}{2}$ (see figure 5.2). Idem for f_π . Under these assumptions, the vector $|\tilde{\Phi}\rangle = U_L U_R |0\rangle$ defined by

$$U_R = e^{i \int d^{d-1}x [\varphi(\bar{x})f_{\varphi,R}(\bar{x}) + \pi(\bar{x})f_{\pi,R}(\bar{x})]} \quad \text{and} \quad U_L = e^{i \int d^{d-1}x [\varphi(\bar{x})f_{\varphi,L}(\bar{x}) + \pi(\bar{x})f_{\pi,L}(\bar{x})]}, \quad (5.14)$$

is a vector representative of the state ϕ for the algebra $\mathcal{A}(\mathcal{W}_{R,\epsilon} \cup \mathcal{W}_{L,\epsilon})$. In fact, the above computation can be done because of the presence of the finite corridor of width ϵ . Moreover, we have that the operator U_R (resp. U_L) acts, by adjoint action, as an automorphism of the algebra of the region $\mathcal{W}_{R,\epsilon}$ (resp. $\mathcal{W}_{L,\epsilon}$), and as the identity transformation over the algebra of the region $\mathcal{W}_{L,\epsilon}$ (resp. $\mathcal{W}_{R,\epsilon}$). Such automorphisms do not change the MI, and according to definition (5.10), we automatically have $\Delta S_{\mathcal{W}_R} = 0$ for these states.

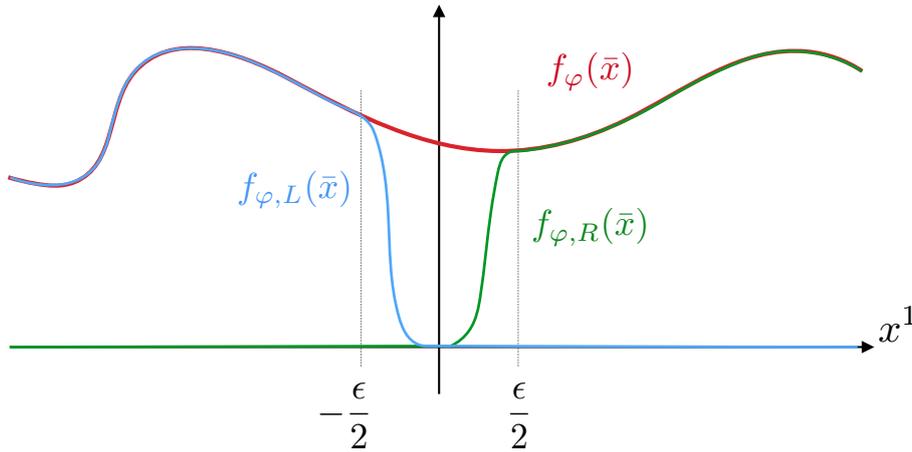


Figure 5.2: For the purpose of the algebra $\mathcal{A}(\mathcal{W}_{R,\epsilon} \cup \mathcal{W}_{L,\epsilon})$, the test function f_φ can be replaced by $f_{\varphi,R} + f_{\varphi,L}$.

¹ $\mathcal{S}(\mathbb{R}^n, \mathbb{R})$ denotes the Schwartz space of real-valued, smooth and exponentially decreasing functions at infinity.

Hence, the question simplifies to see whether for coherent states we have that

$$S_{\mathcal{W}_R}(\phi | \omega_0) = \Delta \langle K_0^R \rangle = 2\pi \int_{x^1 > 0} d^{d-1}x x^1 \langle \Phi | T_{00}(\bar{x}) | \Phi \rangle, \quad (5.15)$$

and which is the right improvement term of the stress tensor. Notice that coherent states can change the expectation value of $:\varphi^2:$. We answer this question in two independent ways:

1. Assuming that (5.15) is correct for some improvement, we show, imposing bounds which come from the positivity and monotonicity of the RE, that the only possibility is the canonical stress tensor, i.e. $\lambda = 0$ (section 5.2).
2. We explicitly compute the relative entropy using Araki formula, and we show that the result (5.15) is correct for the canonical stress tensor (sections 5.3 and 5.5).

5.2. Bounds on boundary terms in the RE

According to the discussion above, there is an ambiguity in the expression (5.15) for the RE of a coherent state coming from the different possible choices of an improving term for the stress energy-momentum tensor. According to (5.9), the relative entropy could be written as the usual contribution with the canonical stress tensor plus a boundary term coming from the improving

$$S(\phi | \omega_0) = \lambda \int_{x_1=0} d^{d-2}x \langle \Phi | \varphi^2(\bar{x}) | \Phi \rangle + 2\pi \int_{x_1>0} d^{d-1}x x^1 \langle \Phi | T_{00}^{can}(\bar{x}) | \Phi \rangle. \quad (5.16)$$

In this section, we assume that this formula is correct, and we show that the only consistent choice is $\lambda = 0$.

A general coherent state can be written as in (5.12) with $f_\varphi, f_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$. In this case, a straightforward computation from (5.16) gives

$$S(\phi | \omega_0) = \lambda \int_{x_1=0} d^{d-2}x f_\pi(\bar{x})^2 + 2\pi \int_{x_1>0} d^{d-1}x \frac{x^1}{2} (f_\varphi(\bar{x})^2 + (\nabla f_\pi(\bar{x}))^2 + m^2 f_\pi(\bar{x})^2). \quad (5.17)$$

Independently of the correct value for λ , if we want that (5.16) and (5.17) represent real expressions for a RE, they must satisfy all the properties known for a RE. In particular we will concentrate on the positivity and the monotonicity (see proposition 3.10). Therefore, the strategy we adopt is to choose functions f_φ and f_π conveniently, and impose positivity and the monotonicity on (5.17) in order to bound the allowed values for λ . In fact, we will show that from the positivity we obtain $\lambda \geq 0$, and from the monotonicity we obtain $\lambda \leq 0$, and hence, it must be that $\lambda = 0$. Then, we conclude that, if we assume that

(5.15) is the correct result for the RE, such an expression only holds for the canonical stress-energy-momentum tensor (5.7).²

Before we continue, we make two simplifications. The first one, which is obvious, is to take $f_\varphi \equiv 0$ and denote $f := f_\pi$. The second one is to work in $d = 1 + 1$ dimensions. The general result for arbitrary dimension could be easily obtained from the former.

5.2.1. Lower bound from positivity

We start with the expression

$$S(\phi | \omega_0) = \lambda f(0)^2 + \pi \int_0^{+\infty} dx x (f'(x)^2 + m^2 f(x)^2), \quad (5.18)$$

where $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$. Then, the positivity of the RE means that

$$0 \leq \lambda f(0)^2 + \pi \int_0^{+\infty} dx x f'(x)^2 + \pi m^2 \int_0^{+\infty} dx x f(x)^2. \quad (5.19)$$

By scaling the function $f(x) \rightarrow f(x/\beta)$ the first two terms of the r.h.s. are constant while the last one gets multiplied by β^2 . Hence, we can make the last term as small as we want and simply take $m = 0$ in the following. Taking f such that $f(0) \neq 0$, we get

$$0 \leq \lambda + \pi \frac{\int_0^{+\infty} dx x f'(x)^2}{f(0)^2}. \quad (5.20)$$

Now, we introduce a convenient family of real functions $(f_a)_{a>0} \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ given by

$$f_a(x) := \log\left(\frac{x}{L} + a\right) e^{-\frac{x}{L}}, \quad x \geq 0, \quad (5.21)$$

where $L > 0$ is a dimensionful fixed constant.³ A straightforward computation shows that the integral in equation (5.20) behaves as

$$\int_0^{+\infty} dx x f_a'(x)^2 = -\log(a) + \mathcal{O}(a^0), \quad 0 < a \ll 1. \quad (5.22)$$

Then, replacing (5.22) into (5.20) we get

$$0 \leq \lambda - \pi \frac{\log(a) + \mathcal{O}(a^0)}{\log^2(a)}. \quad (5.23)$$

²Conversely, it is almost immediate to see that expression (5.17) is positive and monotonic for $\lambda = 0$.

³The functions f_a are smoothly extended to the whole real line. Such an extension is guaranteed by a theorem due to Seeley [112].

Finally, taking the limit $a \rightarrow 0^+$ we get the desired result

$$\lambda \geq 0. \quad (5.24)$$

5.2.2. Upper bound from monotonicity

Monotonicity for the case of the right wedge means

$$S_{\mathcal{W}_{R,y}}(\phi | \omega_0) \geq S_{\mathcal{W}_{R,y'}}(\phi | \omega_0), \quad \text{for any } y' \geq y, \quad (5.25)$$

where $\mathcal{W}_{R,y} := \{x \in \mathbb{R}^d : x^1 > |x^0| + y\}$. In fact, $\mathcal{W}_{R,y}$ is obtained applying a translation of amount y , in the x^1 positive direction, to the right Rindler wedge \mathcal{W}_R . In the following, we denote $S(y) := S_{\mathcal{W}_{R,y}}(\phi | \omega_0)$. Then, we have the expressions

$$S(0) = \lambda f(0)^2 + \pi \int_0^{+\infty} dx x (f'(x)^2 + m^2 f(x)^2), \quad (5.26)$$

$$S(y) = \lambda f(y)^2 + \pi \int_y^{+\infty} dx (x-y) (f'(x)^2 + m^2 f(x)^2), \quad (5.27)$$

where $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$. We can eliminate the mass terms by scaling the test function as in the previous section. The monotonicity $S(0) \geq S(y)$ for $y \geq 0$ reads

$$\lambda (f(y)^2 - f(0)^2) \leq \pi \int_0^y dx x f'(x)^2 + \pi y \int_y^{+\infty} dx f'(x)^2. \quad (5.28)$$

Now, we introduce a convenient family of functions parametrized by the constants $\alpha \in (0, \frac{1}{2})$, $\delta \in (0, 1)$, $y > 0$, $\epsilon > 0$ and given by

$$f_{\alpha,\delta,y,\epsilon}(x) := g_{\alpha,\delta,y}(x) \Theta_{y,\epsilon}(x), \quad \text{for } x \geq 0, \quad (5.29)$$

where

$$g_{\alpha,\delta,y}(x) := \left(\frac{x}{y} (1 - \delta) + \delta \right)^\alpha, \quad (5.30)$$

and $\Theta_{y,\epsilon}$ is a smooth step function satisfying the condition

$$\Theta_{y,\epsilon}(x) = \begin{cases} 1 & x \leq y, \\ 0 & x \geq y + \epsilon. \end{cases} \quad (5.31)$$

We introduce such a step function to ensure that $f_{\alpha,\delta,y,\epsilon} \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ for any values of $(\alpha, \delta, y, \epsilon)$ in the set specified above. The functions $f_{\alpha,\delta,y,\epsilon}$ are smoothly extended for $x \leq 0$ (see footnote 3). The behavior of this function is shown in figure 5.3.

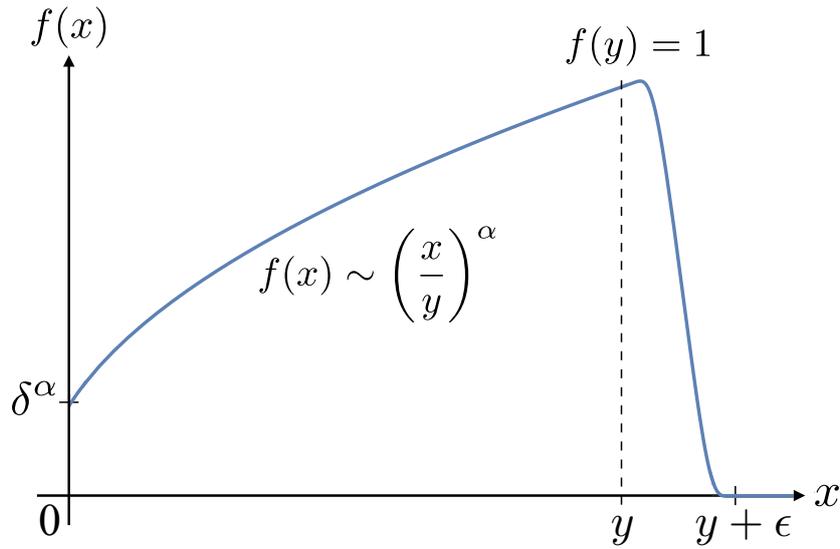


Figure 5.3: A schematic plot of the function $f = f_{\alpha, \delta, y, \epsilon}$.

In particular, we use the following smooth step function

$$\Theta_{y, \epsilon}(x) := \left[1 + \exp \left(-\frac{2\epsilon \left(x - y - \frac{\epsilon}{2} \right)}{\left(x - y - \frac{\epsilon}{2} \right)^2 - \frac{\epsilon^2}{4}} \right) \right]^{-1}, \quad \text{if } y < x < y + \epsilon, \quad (5.32)$$

which has the crucial property $\max_{x \in \mathbb{R}} |\Theta'_{y, \epsilon}(x)| = \frac{2}{\epsilon}$. From now on, to lighten the notation, we do not write the cumbersome subindices of the above functions. We evaluate the different terms of (5.28) independently as

$$f(y)^2 - f(0)^2 = 1 - \delta^{2\alpha}, \quad (5.33)$$

$$\pi \int_0^y dx x f'(x)^2 = \pi \frac{\alpha}{2} \left(\frac{2\alpha\delta - \delta^{2\alpha}}{1 - 2\alpha} + 1 \right), \quad (5.34)$$

$$\begin{aligned} \pi y \int_y^{+\infty} dx f'(x)^2 &\leq \pi y \int_y^{y+\epsilon} dx |g'(x)^2 \Theta(x)^2| + \pi y \int_y^{y+\epsilon} dx |g(x)^2 \Theta'(x)^2| \\ &\quad + \pi y \int_y^{y+\epsilon} dx |2g'(x)g(x)\Theta(x)\Theta'(x)|. \end{aligned} \quad (5.35)$$

We deal with each term of (5.35) separately

$$\begin{aligned} \pi y \int_y^{y+\epsilon} dx |g'(x)^2 \Theta(x)^2| &\leq \pi y \int_y^{y+\epsilon} dx g'(x)^2 = \frac{\pi \alpha^2 (1 - \delta)}{1 - 2\alpha} \left[1 - \left(1 + \frac{(1 - \delta)\epsilon}{y} \right)^{2\alpha - 1} \right] \\ &\xrightarrow{\epsilon \rightarrow +\infty} \frac{\pi \alpha^2 (1 - \delta)}{1 - 2\alpha}, \end{aligned} \quad (5.36)$$

$$\begin{aligned}
\pi y \int_y^{y+\epsilon} dx |2g'(x)g(x)\Theta(x)\Theta'(x)| &\leq \frac{2\pi y}{\epsilon} \int_y^{y+\epsilon} dx 2g'(x)g(x) = \frac{2\pi y}{\epsilon} [g(y+\epsilon)^2 - g(y)^2] \\
&= \frac{2\pi y}{\epsilon} \left[\left(1 + \frac{(1-\delta)\epsilon}{y}\right)^{2\alpha} - 1 \right] \xrightarrow{\epsilon \rightarrow +\infty} 0, \tag{5.37}
\end{aligned}$$

$$\begin{aligned}
\pi y \int_y^{y+\epsilon} dx |g(x)^2\Theta'(x)^2| &\leq \frac{4\pi y}{\epsilon^2} \int_y^{y+\epsilon} dx g(x)^2 \\
&= \frac{4\pi y^2}{(1+2\alpha)(1-\delta)\epsilon^2} \left(\left(1 + \frac{(1-\delta)\epsilon}{y}\right)^{2\alpha+1} - 1 \right) \\
&\xrightarrow{\epsilon \rightarrow +\infty} 0, \tag{5.38}
\end{aligned}$$

where in the last steps of each computation we take the limit $\epsilon \rightarrow +\infty$. It is valid to take this limit in the inequality since it must hold for all $\epsilon > 0$. Replacing these partial results on (5.28) we arrive to

$$\lambda(1 - \delta^{2\alpha}) \leq \pi \frac{\alpha}{2} \left(\frac{2\alpha\delta - \delta^{2\alpha}}{1 - 2\alpha} + 1 \right) + \frac{\pi\alpha^2(1-\delta)}{1 - 2\alpha}. \tag{5.39}$$

Then, taking the limit $\delta \rightarrow 0^+$ we get

$$\lambda \leq \pi \frac{\alpha}{2} + \pi \frac{\alpha^2}{1 - 2\alpha}, \tag{5.40}$$

and finally, taking $\alpha \rightarrow 0^+$ we arrive to the desired result

$$\lambda \leq 0. \tag{5.41}$$

5.3. Algebraic theory of the free real scalar field

In this section, we briefly review the algebraic theory of the free real scalar field. Concretely, we show explicitly the definition of the net of local algebras, satisfying all the axioms of the definitions 2.56 and 2.58. The material of this section is broadly discussed in [59, 113–115]. For our convenience, we introduce our own notation, and we develop some relations between the concepts, which, to our opinion, were not very clearly discussed in these references.

5.3.1. Local algebras for spacetime regions

The algebraic theory of the free real scalar field is defined as a net of vN algebras acting on the Fock Hilbert space \mathcal{H}_0 . In fact, this net defines the vacuum representation of the

theory according to definition 2.58. The Hilbert space \mathcal{H}_0 is constructed as the symmetric tensor product of the one-particle Hilbert space. To describe it properly we introduce the following three vector spaces.

The space of test functions. The space of test functions is the Schwartz space $\mathcal{S}(\mathbb{R}^d, \mathbb{R})$ of real, smooth and exponentially decreasing functions at infinity. This space carries naturally a representation of the restricted Poincaré group \mathcal{P}_+^\uparrow given by $f \mapsto f_{(\Lambda, a)}$, with $f_{(\Lambda, a)}(x) := f(\Lambda(x - a))$ for any $(\Lambda, a) \in \mathcal{P}_+^\uparrow$.

The one particle Hilbert space. The Hilbert space \mathfrak{H} of one-particle states of mass $m > 0$ and zero spin is made up of the square-integrable functions on the mass shell hyperboloid $H_m := \{p \in \mathbb{R}^d : p^2 = m^2, p^0 > 0\}$ with the Poincaré invariant measure $d\mu(p) := \Theta(p^0)\delta(p^2 - m^2)d^d p$. It can be realized as

$$\mathfrak{H} = L^2\left(\mathbb{R}^{d-1}, \frac{d^{d-1}p}{2\omega(\bar{p})}\right), \quad (5.42)$$

$$\langle \mathbf{f} | \mathbf{g} \rangle_{1p} = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega(\bar{p})} \mathbf{f}(\bar{p})^* \mathbf{g}(\bar{p}), \quad (5.43)$$

where $p^0 = \sqrt{\bar{p}^2 + m^2} =: \omega(\bar{p})$.⁴ Such a space carries a unitary representation of \mathcal{P}_+^\uparrow given by $(u(\Lambda, a)\mathbf{f})(p) := e^{ip \cdot a} \mathbf{f}(\Lambda^{-1}p)$ for any $\mathbf{f} \in \mathfrak{H}$ and $(\Lambda, a) \in \mathcal{P}_+^\uparrow$.

The Fock Hilbert space. The Fock Hilbert space \mathcal{H}_0 is the direct sum of the symmetric tensor powers of the one-particle Hilbert space \mathfrak{H}

$$\mathcal{H}_0 := \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n, sym}. \quad (5.44)$$

For each $\mathbf{h} \in \mathfrak{H}$, the creation and annihilation operators $A^*(\mathbf{h})$ and $A(\mathbf{h})$ act on \mathcal{H} as usual

$$A^*(\mathbf{h}) \text{sym}(\mathbf{h}_1 \otimes \cdots \otimes \mathbf{h}_n) := \sqrt{n+1} \text{sym}(\mathbf{h} \otimes \mathbf{h}_1 \otimes \cdots \otimes \mathbf{h}_n), \quad (5.45)$$

$$A(\mathbf{h}) \text{sym}(\mathbf{h}_1 \otimes \cdots \otimes \mathbf{h}_n) := \frac{1}{\sqrt{n}} \sum_{j=1}^n \langle \mathbf{h}_j | \mathbf{h} \rangle_{1p} \text{sym}(\mathbf{h}_1 \otimes \cdots \otimes \widehat{\mathbf{h}}_j \otimes \cdots \otimes \mathbf{h}_n), \quad (5.46)$$

where $\widehat{\mathbf{h}}_j$ means that the vector is excluded from the tensor product on the r.h.s. of (5.46). The Fock space naturally inherits from \mathfrak{H} a unitary representation of \mathcal{P}_+^\uparrow , which is denoted by $U(\Lambda, a)$. According to that, there is a unique (up to a phase) \mathcal{P}_+^\uparrow -invariant vector

⁴We denote by $\langle \cdot | \cdot \rangle_{1p}$ the inner product in the one-particle Hilbert space \mathfrak{H} , whereas $\langle \cdot | \cdot \rangle$ denotes the inner product in the Fock Hilbert space \mathcal{H}_0 .

denoted by $|0\rangle := \mathbf{1} \in \mathfrak{H}^{\otimes 0}$, which is called the *vacuum vector*. For each $\mathbf{h} \in \mathfrak{H}$, the normalized vector

$$e^{\mathbf{h}} := e^{-\frac{1}{2}\|\mathbf{h}\|_{1p}^2} \sum_{n=0}^{\infty} \frac{\mathbf{h}^{\otimes n}}{\sqrt{n!}} \in \mathcal{H}_0, \quad (5.47)$$

is called *coherent vector*, and it satisfies the relations $e^{\mathbf{0}} = |0\rangle$ and $\langle 0|e^{\mathbf{h}}\rangle = e^{-\frac{1}{2}\|\mathbf{h}\|_{1p}^2}$.

It is a well-known fact that the structure of a free QFT is completely determined by the underlying one-particle quantum theory. More concretely, the assignment $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ is defined by the composition of two different maps

$$\mathcal{O} \in \mathcal{K} \mapsto K(\mathcal{O}) \subset \mathfrak{H}, \quad (5.48)$$

$$K \subset \mathfrak{H} \mapsto \mathcal{A}(K) \subset \mathcal{B}(\mathcal{H}_0), \quad (5.49)$$

which are called 1st and 2nd quantization maps respectively.⁵ We will explain each map separately.

5.3.1.1. First quantization map

For any closed linear subspace $K \subset \mathfrak{H}$ we define its *symplectic complement* as

$$K' := \left\{ \mathbf{h} \in \mathfrak{H} : \langle \mathbf{h} | \mathbf{k} \rangle_{1p} = 0, \text{ for all } \mathbf{k} \in K \right\}. \quad (5.50)$$

Now, we consider the following real dense embedding $E : \mathcal{S}(\mathbb{R}^d, \mathbb{R}) \rightarrow \mathfrak{H}$

$$(Ef)(\bar{p}) := \sqrt{2\pi} \hat{f} |_{H_m}(\bar{p}) = \sqrt{2\pi} \hat{f}(\omega(\bar{p}), \bar{p}), \quad (5.51)$$

where $\hat{f}(p) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{ip \cdot x} d^d x$ is the usual Fourier transform. Such an embedding is Poincaré covariant, i.e. $E(f_{(\Lambda, a)}) = u(\Lambda, a) E(f)$. From now on, we naturally identified functions on $\mathcal{S}(\mathbb{R}^d, \mathbb{R})$ with vectors on \mathfrak{H} through the above embedding.

The *1st quantization map* is an assignment

$$\mathcal{O} \in \mathcal{K} \mapsto K(\mathcal{O}) \subset \mathfrak{H}, \quad (5.52)$$

where $K(\mathcal{O})$ is a real closed linear subspace defined as

$$K(\mathcal{O}) := \overline{\{E(f) : f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}) \text{ and } \text{supp}(f) \subset \mathcal{O}\}} \subset \mathfrak{H}. \quad (5.53)$$

It is not difficult to see that such a map satisfies the duality $K(\mathcal{O}') = K(\mathcal{O})'$ [114].

⁵ \mathcal{K} denotes the set of causally complete regions of the spacetime \mathbb{R}^d (see definition 2.52).

5.3.1.2. Second quantization map

We define the embedding $W : \mathfrak{H} \rightarrow \mathcal{B}(\mathcal{H}_0)$

$$W(\mathbf{h}) := e^{i(A(\mathbf{h}) + A^*(\mathbf{h}))}. \quad (5.54)$$

The operators $W(\mathbf{h})$ are called *Weyl unitaries*. These operators satisfy the canonical commutation relations (CCR) in the Segal's form [113]

$$W(\mathbf{h}_1) W(\mathbf{h}_2) = e^{-i \operatorname{Im} \langle \mathbf{h}_1, \mathbf{h}_2 \rangle_{\mathbb{P}}} W(\mathbf{h}_1 + \mathbf{h}_2), \quad (5.55)$$

$$W(\mathbf{h})^* = W(-\mathbf{h}). \quad (5.56)$$

A Poincaré unitary $U(\Lambda, a)$ acts covariantly on a Weyl operator according to

$$U(\Lambda, a) W(\mathbf{h}) U(\Lambda, a)^* = W(u(\Lambda, a)\mathbf{h}), \quad (5.57)$$

$$W(\mathbf{h})|0\rangle = e^{i\mathbf{h}}. \quad (5.58)$$

The *2nd quantization map* is an assignment

$$K \subset \mathfrak{H} \mapsto \mathcal{A}(K) \subset \mathcal{B}(\mathcal{H}), \quad (5.59)$$

from the set of real closed linear subspaces of \mathfrak{H} to the set of vN subfactors of $\mathcal{B}(\mathcal{H}_0)$. It is defined by means of

$$\mathcal{A}(K) := \{W(\mathbf{k}) : \mathbf{k} \in K\}'' \subset \mathcal{B}(\mathcal{H}_0). \quad (5.60)$$

This map satisfies the duality $\mathcal{A}(K') = \mathcal{A}(K)'$ [113].

5.3.1.3. Net of local algebras

According to the previous discussion, the net of local algebras $\mathcal{O} \in \mathcal{K} \rightarrow \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0)$ of the free real scalar field is defined as the composition of the 1st and 2nd quantization maps, i.e.

$$\mathcal{A}(\mathcal{O}) := \mathcal{A}(K(\mathcal{O})). \quad (5.61)$$

This net satisfies all the axioms of definitions 2.56, 2.58, and 2.61. Moreover, the map $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ in (5.61) satisfies the properties 1, 2', and 3-5 of definition 2.75 [114]. For any $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, the field operator $\phi(f)$ is defined through the relation

$$W(E(f)) = e^{i\phi(f)} =: W(f). \quad (5.62)$$

5.3.2. Local algebras at a fixed time

In this subsection, we discuss the theory of the vN algebras for the free real scalar field at a fixed time $x^0 = 0$. Naively speaking, they are the local algebras generated by the field operator at a fixed time $\varphi(\bar{x})$ and its canonical conjugate momentum field $\pi(\bar{x})$. This approach is very useful for the computation of the RE in section 5.5.

We can decompose $\mathfrak{H} = \mathfrak{H}_\varphi \oplus_{\mathbb{R}} \mathfrak{H}_\pi$ into two \mathbb{R} -linear closed subspaces

$$\mathfrak{H}_\varphi := \{ \mathbf{h} \in \mathfrak{H} : \mathbf{h}(\bar{p}) = \mathbf{h}(-\bar{p})^* \text{ a.e.} \}, \quad (5.63)$$

$$\mathfrak{H}_\pi := \{ \mathbf{h} \in \mathfrak{H} : \mathbf{h}(\bar{p}) = -\mathbf{h}(-\bar{p})^* \text{ a.e.} \}. \quad (5.64)$$

Each $\mathbf{h} \in \mathfrak{H}$ can be uniquely written as $\mathbf{h} = \mathbf{h}_\varphi + \mathbf{h}_\pi$, where

$$\mathbf{h}_\varphi(\bar{p}) := \frac{\mathbf{h}(\bar{p}) + \mathbf{h}(-\bar{p})^*}{2} \quad \text{and} \quad \mathbf{h}_\pi(\bar{p}) := \frac{\mathbf{h}(\bar{p}) - \mathbf{h}(-\bar{p})^*}{2}. \quad (5.65)$$

We also have the following useful relations

$$\text{Im} \langle \mathbf{h}_\varphi | \mathbf{h}'_\varphi \rangle_{1\text{p}} = \text{Im} \langle \mathbf{h}_\pi | \mathbf{h}'_\pi \rangle_{1\text{p}} = \text{Re} \langle \mathbf{h}_\varphi | \mathbf{h}'_\pi \rangle_{1\text{p}} = 0, \quad (5.66)$$

for all $\mathbf{h}_\varphi, \mathbf{h}'_\varphi \in \mathfrak{H}_\varphi$ and $\mathbf{h}_\pi, \mathbf{h}'_\pi \in \mathfrak{H}_\pi$.

Now, we consider the following real dense embeddings $E_{\varphi,\pi} : \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \rightarrow \mathfrak{H}_{\varphi,\pi}$

$$(E_\varphi f)(\bar{p}) := \hat{f}(\bar{p}) \quad \text{and} \quad (E_\pi f)(\bar{p}) := i\omega(\bar{p}) \hat{f}(\bar{p}), \quad (5.67)$$

where $\hat{f}(\bar{p}) := (2\pi)^{-\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} f(\bar{x}) e^{-i\bar{p}\cdot\bar{x}} d^{d-1}x$. From now on, we naturally identify functions on $\mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ with vectors on $\mathfrak{H}_\varphi, \mathfrak{H}_\pi$ through these embeddings. The map E_φ (resp. E_π) is actually defined on a bigger class of test functions, namely $H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})$ (resp. $H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})$), i.e.

$$E_\varphi : H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \rightarrow \mathfrak{H}_\varphi \quad \text{and} \quad E_\pi : H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \rightarrow \mathfrak{H}_\pi, \quad (5.68)$$

where $H^\alpha(\mathbb{R}^{d-1}, \mathbb{R})$ is the real Sobolev space of order α (see appendix B). We have that $E_\varphi(H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})) = \mathfrak{H}_\varphi$ and $E_\pi(H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})) = \mathfrak{H}_\pi$. For any $\mathbf{h}_\varphi \in \mathfrak{H}_\varphi$ and $\mathbf{h}_\pi \in \mathfrak{H}_\pi$, we use the map (5.54) to define the Weyl unitaries

$$W_\varphi(\mathbf{h}_\varphi) := W(\mathbf{h}_\varphi) \quad \text{and} \quad W_\pi(\mathbf{h}_\pi) := W(\mathbf{h}_\pi), \quad (5.69)$$

which satisfy the CCR in the Weyl form [113]

$$W_\varphi(\mathbf{h}_\varphi) W_\pi(\mathbf{h}_\pi) W_\varphi(\mathbf{h}'_\varphi) W_\pi(\mathbf{h}'_\pi) = W_\varphi(\mathbf{h}_\varphi + \mathbf{h}'_\varphi) W_\pi(\mathbf{h}_\pi + \mathbf{h}'_\pi) e^{2i \operatorname{Im} \langle \mathbf{h}'_\varphi | \mathbf{h}_\pi \rangle_{1p}}, \quad (5.70)$$

$$W_\varphi(\mathbf{h}_\varphi)^* = W_\varphi(-\mathbf{h}_\varphi), \quad (5.71)$$

$$W_\pi(\mathbf{h}_\pi)^* = W_\pi(-\mathbf{h}_\pi). \quad (5.72)$$

The *field operator at a fixed time* $\varphi(f_\varphi)$ and its *canonical conjugate momentum field* $\pi(f_\pi)$ are defined by the formulas

$$W_\varphi(E_\varphi(f_\varphi)) = e^{i\varphi(f_\varphi)} =: W_\varphi(f_\varphi) \quad \text{and} \quad W_\pi(E_\pi(f_\pi)) =: e^{i\pi(f_\pi)} =: W_\pi(f_\pi). \quad (5.73)$$

Here again, the local algebras at a fixed time are also defined through the 1st and 2nd quantization maps.

5.3.2.1. First quantization map

We say that $\mathcal{C} \subset \mathbb{R}^{d-1}$ is a *spatially complete* region if it is open, regular,⁶ and with regular boundary.⁷ The set of all spatially complete regions is denoted by $\tilde{\mathcal{K}}_0$, which is a proper subset of the set of all open regions \mathcal{K}_0 .⁸ From now on, we only consider spatially complete regions.

The *1st quantization map*

$$\mathcal{C} \in \tilde{\mathcal{K}}_0 \mapsto (K_\varphi(\mathcal{C}), K_\pi(\mathcal{C})) \subset (\mathfrak{H}_\varphi, \mathfrak{H}_\pi), \quad (5.74)$$

is defined by means

$$K_\varphi(\mathcal{C}) := \overline{\{E_\varphi(f) : f \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \text{ and } \operatorname{supp}(f) \subset \mathcal{C}\}} \subset \mathfrak{H}_\varphi, \quad (5.75)$$

$$K_\pi(\mathcal{C}) := \overline{\{E_\pi(f) : f \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R}) \text{ and } \operatorname{supp}(f) \subset \mathcal{C}\}} \subset \mathfrak{H}_\pi. \quad (5.76)$$

It can be shown that [114]

$$K_\varphi(\mathcal{C}) = \left\{ E_\varphi(f) : f \in H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \text{ and } \operatorname{supp}(f) \subset \bar{\mathcal{C}} \text{ a.e.} \right\}, \quad (5.77)$$

$$K_\pi(\mathcal{C}) = \left\{ E_\pi(f) : f \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \text{ and } \operatorname{supp}(f) \subset \bar{\mathcal{C}} \text{ a.e.} \right\}. \quad (5.78)$$

⁶An open set $U \subset \mathbb{R}^n$ is *regular* iff $U \equiv \operatorname{Int}(\bar{U})$.

⁷We say that $\mathcal{C} \subset \mathbb{R}^{d-1}$ has a *regular boundary* if $\partial\mathcal{C} \subset \mathbb{R}^{d-1}$ is a smooth submanifold of dimension $d-2$, or several manifolds joined together along smooth manifolds of dimension $d-3$ [114].

⁸According to proposition 2.81, the set of all open regions \mathcal{K}_0 is an orthocomplemented lattice. It can be shown that $\tilde{\mathcal{K}}_0$ is a sublattice of \mathcal{K}_0 [59, 114].

5.3.2.2. Second quantization map

Let $K_\varphi \subset \mathfrak{H}_\varphi$ and $K_\pi \subset \mathfrak{H}_\pi$ be \mathbb{R} -linear closed subspaces. The 2nd quantization map

$$(K_\varphi, K_\pi) \subset (\mathfrak{H}_\varphi, \mathfrak{H}_\pi) \mapsto \mathcal{A}_0(K_\varphi, K_\pi) \subset \mathcal{B}(\mathcal{H}_0), \quad (5.79)$$

is defined by means

$$\mathcal{A}_0(K_\varphi, K_\pi) := \{W_\varphi(k_\varphi)W_\pi(k_\pi) : k_\varphi \in K_\varphi, k_\pi \in K_\pi\}'' \subset \mathcal{B}(\mathcal{H}_0). \quad (5.80)$$

5.3.2.3. Net of local algebras at a fixed time

The net of local algebras $\mathcal{C} \in \tilde{\mathcal{K}}_0 \mapsto \mathcal{A}_0(\mathcal{C}) \subset \mathcal{B}(\mathcal{H}_0)$ at a fixed time is defined as the composition of the 1st and 2nd quantization maps, i.e.

$$\mathcal{A}_0(\mathcal{C}) := \mathcal{A}_0(K_\varphi(\mathcal{C}), K_\pi(\mathcal{C})). \quad (5.81)$$

The above net satisfies all the properties of proposition 2.84 [114]. In other words, it is an homomorphism of orthocomplemented lattices.

5.3.3. Relation between the two approaches

In this subsection, we explain the relation existing between the two approaches of sections 5.3.1 and 5.3.2.

The relation between nets. Given a spatially complete region $\mathcal{C} \in \tilde{\mathcal{K}}_0$, we denote its Cauchy development as $\mathcal{O}_\mathcal{C} := D(\mathcal{C}) \in \mathcal{K}$. Then, the following relation holds [116]

$$K(\mathcal{O}_\mathcal{C}) = K_\varphi(\mathcal{C}) \oplus_{\mathbb{R}} K_\pi(\mathcal{C}), \quad (5.82)$$

and hence, we have the following equality of vN algebras

$$\mathcal{A}_0(\mathcal{C}) = \mathcal{A}(\mathcal{O}_\mathcal{C}). \quad (5.83)$$

The relations developed along the above subsections can be summarized in the following schematic diagram

$$\begin{array}{ccccc} \mathcal{O}_\mathcal{C} \in \mathcal{K} & \xrightarrow{E} & K \subset \mathfrak{H} & \xrightarrow{W} & \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \\ \uparrow D & & \uparrow \oplus_{\mathbb{R}} & & \parallel \\ \mathcal{C} \in \tilde{\mathcal{K}}_0 & \xrightarrow{E_{\varphi, \pi}} & (K_\varphi, K_\pi) \subset \mathfrak{H}_\varphi \oplus_{\mathbb{R}} \mathfrak{H}_\pi & \xrightarrow{W_{\varphi, \pi}} & \mathcal{A}_0 \subset \mathcal{B}(\mathcal{H}). \end{array} \quad (5.84)$$

The relation between test functions. Given any $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, we can define

$$F(x) := \int_{\mathbb{R}^d} \Delta(x-y) f(y) d^d y, \quad (5.85)$$

where⁹

$$\Delta(x) := -i(2\pi)^{-(d-1)} \int_{\mathbb{R}^d} e^{-ip \cdot x} \delta(p^2 - m^2) \operatorname{sgn}(p^0) d^d p. \quad (5.86)$$

It follows that F is a solution of the Klein-Gordon equation, i.e. $(\square + m^2)F = 0$, and whereby, we can take its initial Cauchy data at $x^0 = 0$ through

$$f_\varphi(\bar{x}) := -\frac{\partial F}{\partial x^0}(0, \bar{x}) \quad \text{and} \quad f_\pi(\bar{x}) := F(0, \bar{x}). \quad (5.87)$$

Finally, it can be shown that $f_\varphi, f_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ and

$$E(f) = E_\varphi(f_\varphi) + E_\pi(f_\pi). \quad (5.88)$$

Moreover, since $F(x) = 0$ if $x \in \operatorname{supp}(f)'$, then we have that

$$\operatorname{supp}(f) \subset \mathcal{O}_C \Rightarrow \operatorname{supp}(f_\varphi), \operatorname{supp}(f_\pi) \subset \mathcal{C}. \quad (5.89)$$

The relation between Weyl unitaries. For the particular case of Weyl unitaries, it follows that

$$W(f) = e^{i\operatorname{Im}\langle f_\varphi | f_\pi \rangle_{1\mathbb{P}}} W_\varphi(f_\varphi) W_\pi(f_\pi), \quad (5.90)$$

where

$$\operatorname{Im}\langle f_\varphi, f_\pi \rangle_{\mathfrak{H}} = \frac{1}{2} \int_{\mathbb{R}^{d-1}} f_\varphi(\bar{x}) f_\pi(\bar{x}) d^{d-1}x. \quad (5.91)$$

5.4. Symmetry of the RE for coherent states

In this section, we study some general relations about the RE for coherent states, which are valid for any region $\mathcal{O} \in \mathcal{K}$. In particular, we show that the RE for coherent states is symmetric.

Notation 5.1. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra and $|\Omega\rangle, |\Phi\rangle \in \mathcal{H}$ be standard vectors. We denote by S_Φ and Δ_Φ the modular involution and the modular operator associated with $\{\mathcal{A}, |\Phi\rangle\}$, and by $S_{\Phi, \Omega}$ and $\Delta_{\Phi, \Omega}$ the relative modular involution and the relative modular operator associated with $\{\mathcal{A}, |\Omega\rangle, |\Phi\rangle\}$.

Coherent states come from acting with a Weyl operator on the vacuum vector. Weyl unitaries have the very important property that they implement, by adjoint action, au-

⁹In the Wightman approach to QFT, we recognize $\Delta(x-y) := -i[\phi(x), \phi(y)]$.

tomorphisms for any local algebra $\mathcal{A}(\mathcal{O})$. More concretely, for any $\mathbf{h} \in \mathfrak{H}$ and any Weyl operator $W(f) \in \mathcal{A}(\mathcal{O})$ ($\text{supp}(f) \subset \mathcal{O}$) we have that

$$W(\mathbf{h})^* W(f) W(\mathbf{h}) = e^{2i\text{Im}\langle f|\mathbf{h}\rangle_{\text{ip}}} W(f) \in \mathcal{A}(\mathcal{O}), \quad (5.92)$$

which implies that $W(\mathbf{h})^* \mathcal{A}(\mathcal{O}) W(\mathbf{h}) = \mathcal{A}(\mathcal{O})$. This property has interesting implications on the RE itself. Indeed, it implies that the relative entropy between a coherent state and the vacuum is symmetric. To show this property, we first prove the following lemmas.

Lemma 5.2. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a vN algebra, $|\Omega\rangle, |\Phi\rangle \in \mathcal{H}_0$ standard vectors, and $U \in \mathcal{B}(\mathcal{H})$ unitary such that $U^* \mathcal{A} U = \mathcal{A}$. Then,*

1. $U|\Omega\rangle$ and $U|\Phi\rangle$ are standard vectors.
2. $S_{U\Omega} = US_{\Omega}U^* \Rightarrow \Delta_{U\Omega} = U\Delta_{\Omega}U^*$.
3. $S_{U\Omega, U\Phi} = US_{\Omega, \Phi}U^* \Rightarrow \Delta_{U\Omega, U\Phi} = U\Delta_{\Omega, \Phi}U^*$.

Proof. (1) $\overline{\mathcal{A}U|\Omega\rangle} = \overline{U\mathcal{A}|\Omega\rangle} = U\overline{\mathcal{A}|\Omega\rangle} = \mathcal{H}$ implies that $U|\Omega\rangle$ is cyclic. Then, $AU|\Omega\rangle = 0 \Leftrightarrow U^*AU|\Omega\rangle = 0 \Leftrightarrow U^*AU = 0 \Leftrightarrow A = 0$ implies that $U|\Omega\rangle$ is separating. The same holds for $U|\Phi\rangle$. (2) For any $A \in \mathcal{A}$, we have that $(US_{\Omega}U^*)AU|\Omega\rangle = US_{\Omega}(U^*AU)|\Omega\rangle = U(U^*AU)^*|\Omega\rangle = A^*U|\Omega\rangle$. Then, applying the polar decomposition, we have that $\Delta_{U\Omega} = U\Delta_{\Omega}U^*$. (3) For any $A \in \mathcal{A}$, we have that $(US_{\Omega, \Phi}U^*)AU|\Phi\rangle = US_{\Omega, \Phi}(U^*AU)|\Phi\rangle = U(U^*AU)^*|\Phi\rangle = A^*U|\Phi\rangle$. Then, $\Delta_{U\Omega, U\Phi} = U\Delta_{\Omega, \Phi}U^*$ follows from applying the polar decomposition. \square

Given a state ω on a vN algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and a unitary $U \in \mathcal{B}(\mathcal{H})$, we denote by ω_U the state defined by $\omega_U(\cdot) := \omega(U^* \cdot U)$.

Lemma 5.3. *Given $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ a vN algebra in standard form, ω a faithful normal state and $U \in \mathcal{B}(\mathcal{H})$ unitary such that $U^* \mathcal{A} U = \mathcal{A}$. Then*

$$S_{\mathcal{A}}(\omega_U | \omega) = S_{\mathcal{A}}(\omega | \omega_{U^*}). \quad (5.93)$$

Proof. Let $|\Omega\rangle$ be a standard vector representative of ω . Then $U|\Omega\rangle$ and $U^*|\Omega\rangle$ are the vector representatives of ω_U and ω_{U^*} , and they are standard vectors because of 1 in lemma 5.2. Using 3 of the same lemma we have that $S_{\Omega, U\Omega} = S_{U^*\Omega, \Omega} = US_{U^*\Omega, \Omega}U^*$. Then, $S(\omega_U | \omega) = \langle U\Omega | K_{\Omega, U\Omega} | U\Omega \rangle = \langle U\Omega | UK_{U^*\Omega, \Omega}U^* | U\Omega \rangle = \langle \Omega | K_{U^*\Omega, \Omega} | \Omega \rangle = S(\omega | \omega_{U^*})$. \square

Now, we come back to coherent states. From now on $\omega_0(\cdot) := \langle 0 | \cdot | 0 \rangle$ denotes the vacuum state. Given any $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, we define the coherent state $\omega_f(\cdot) := \langle 0 | W(f)^* \cdot W(f) | 0 \rangle$. The Reeh-Schlieder theorem 2.62 asserts that the vacuum vector $|0\rangle$ is standard for any

local algebra $\mathcal{A}(\mathcal{O})$ of any standard region $\mathcal{O} \in \mathcal{K}$. Then, lemma 5.2 ensures the same for the coherent vector $W(f)|0\rangle$. Then, lemma 5.3 implies

$$S_{\mathcal{O}}(\omega_f | \omega_0) = S_{\mathcal{O}}(\omega_0 | \omega_{-f}) , \quad (5.94)$$

for any coherent state ω_f and any standard region $\mathcal{O} \in \mathcal{K}$.

Moreover, the net algebra of the free scalar field has a global \mathbb{Z}_2 -symmetry implemented by an operator $z = z^{-1} = z^*$ such that¹⁰

$$zW(f)z := W(-f) = W(f)^* , \quad \text{and} \quad z|0\rangle = |0\rangle . \quad (5.95)$$

This motivates the following lemma.

Lemma 5.4. *For the local algebra $\mathcal{A}(\mathcal{O})$ of any standard region $\mathcal{O} \in \mathcal{K}$, the RE between a coherent state ω_f and the vacuum state ω_0 satisfies*

$$S(\omega_f | \omega_0) = S(\omega_{-f} | \omega_0) . \quad (5.96)$$

Proof. Let $|0\rangle$, $W(f)|0\rangle$, and $W(f)^*|0\rangle$ be the vector representatives of the states ω_0 , ω_f , and ω_{-f} . Let us denote by $S_{0,f}$ the relative modular involution associated with $\{\mathcal{A}(\mathcal{O}), W(f)|0\rangle, |0\rangle\}$. Then, employing the \mathbb{Z}_2 -symmetry (5.95), we have that

$$(zS_{0,f}z)W(g)W(f)^*|0\rangle = zS_{0,f}W(g)^*W(f)|0\rangle = zW(g)|0\rangle = W(g)^*|0\rangle , \quad (5.97)$$

for all $W(g) \in \mathcal{A}(\mathcal{O})$. Then, $S_{0,-f} = zS_{0,f}z$, and hence, $K_{0,-f} = zK_{0,f}z$. Finally, we get $S(\omega_f | \omega_0) = \langle 0|K_{0,f}|0\rangle = \langle 0|K_{0,-f}|0\rangle = S(\omega_{-f} | \omega_0)$. \square

Remark 5.5. The above lemma should apply to any scalar field theory with \mathbb{Z}_2 -symmetry as above, as long as it satisfies the Garding-Wightman axioms.

Finally, combining (5.94) and (5.96), we have the following theorem concerning the symmetry for the relative entropy between coherent states.

Theorem 5.6. *For the local algebra $\mathcal{A}(\mathcal{O})$ of any standard region $\mathcal{O} \in \mathcal{K}$, the RE between a coherent state ω_f and the vacuum state ω is symmetric, i.e.*

$$S(\omega_f | \omega_0) = S(\omega_0 | \omega_f) . \quad (5.98)$$

To end, we have the following theorem concerning the relative entropy between two coherent states.¹¹

¹⁰In the Lagrangian approach to QFT, this is the usual symmetry $\phi(x) \rightarrow -\phi(x)$.

¹¹This result has been found in the past using other methods. For example, see [117] for a derivation using the replica trick for $2d$ free CFTs.

Theorem 5.7. *For the local algebra $\mathcal{A}(\mathcal{O})$ of any standard region $\mathcal{O} \in \mathcal{K}$, the RE between two coherent states ω_f and ω_g satisfies*

$$S(\omega_f | \omega_g) = S(\omega_{f-g} | \omega_0). \quad (5.99)$$

Proof. Let $|0\rangle$, $W(f)|0\rangle$, and $W(g)|0\rangle$ be the vector representatives of the states ω_0 , ω_f , and ω_g . Let us denote by $S_{g,f}$ the relative modular involution associated with $\{\mathcal{A}(\mathcal{O}), W(f)|0\rangle, W(g)|0\rangle\}$. Then, using 3 in lemma 5.2, we have that $W(g)^* S_{g,f} W(g)$ is the relative modular involution associated with $\{\mathcal{A}(\mathcal{O}), W(g)^* W(f)|0\rangle, |0\rangle\}$. Since $W(g)^* W(f)|0\rangle$ is a vector representative of ω_{f-g} , we have that $S(\omega_f | \omega_g) = \langle 0 | W(f)^* S_{g,f} W(f) | 0 \rangle = S(\omega_{f-g} | \omega_0)$. \square

5.5. Relative entropy for the Rindler wedge

In this section, we compute the RE, in the theory of the free real scalar field, between a coherent state and the vacuum using Araki formula (definition 3.31). We focus on the case of the Rindler wedge region. The aim is to show explicitly that formula (5.15) holds for the canonical stress-tensor.

Let $\mathcal{A}_{\mathcal{W}} := \mathcal{A}(\mathcal{W}_R)$ be the right Rindler wedge algebra and $\mathcal{A}_{\mathcal{W}'} := \mathcal{A}(\mathcal{W}'_R)$. We define $\Sigma_R := \{\bar{x} \in \mathbb{R}^{d-1} : x^1 > 0\}$. Then, we have that $\mathcal{W}_R = D(\Sigma_R)$ and $\mathcal{W}_L := \mathcal{W}'_R = D(\Sigma'_R)$, and hence, we have that

$$\mathcal{A}_0(\Sigma_R) = \mathcal{A}_{\mathcal{W}} \quad \text{and} \quad \mathcal{A}_0(\Sigma'_R) = \mathcal{A}_{\mathcal{W}'}. \quad (5.100)$$

Let ω_0 be the vacuum state and ω_f a coherent state with $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$. Let us call $|0\rangle$ and $|\Phi\rangle := W(f)|0\rangle$ its vector representatives. In order to compute the relative entropy $S(\omega_f | \omega_0)$, we need to calculate the relative modular Hamiltonian $K_{0,\Phi}$, or $K_{\Phi,0}$ because of theorem 5.6. As we have explained in the last subsection, the vectors $|0\rangle$ and $|\Phi\rangle$ are standard for $\mathcal{A}_{\mathcal{W}}$. We distinguish between two cases

$$\text{easy case} : f = f_L + f_R, \quad (5.101)$$

$$\text{hard case} : f \neq f_L + f_R, \quad (5.102)$$

where $\text{supp}(f_L) \in \mathcal{W}_L$ and $\text{supp}(f_R) \in \mathcal{W}_R$.¹² In the following subsections, we deal with each case (5.101) and (5.102) separately.

¹²In particular, the easy case includes the cases when $W(f) \in \mathcal{A}_{\mathcal{W}}$ or $W(f) \in \mathcal{A}_{\mathcal{W}'}$.

5.5.1. Easy case: $f = f_L + f_R$

In this case, we have that the coherent vector can be written as $W(f) = W(f_L)W(f_R)$ with $W(f_L) \in \mathcal{A}_{\mathcal{W}'}$ and $W(f_R) \in \mathcal{A}_{\mathcal{W}}$. This case can be solved using the following lemma.

Lemma 5.8. *Given $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ a vN algebra, $|\Omega\rangle$ a standard vector and $U \in \mathcal{A}$ and $U' \in \mathcal{A}'$ unitaries. Then $|\Phi\rangle = U'U|\Omega\rangle$ is standard and*

$$S_{\Omega, \Phi} = US_{\Omega}U'^*. \quad (5.103)$$

Then, by polar decomposition we have that $J_{\Omega, \Phi} = UJ_{\Omega}U'^*$, $\Delta_{\Omega, \Phi} = U'\Delta_{\Omega}U'^*$ and $K_{\Omega, \Phi} = U'K_{\Omega}U'^*$.

Proof. $\overline{\mathcal{A}|\Phi\rangle} = \overline{\mathcal{A}U'U|\Omega\rangle} = U'\overline{\mathcal{A}U|\Omega\rangle} = U'\overline{\mathcal{A}_{\mathcal{W}}|\Omega\rangle} = \overline{\mathcal{A}|\Omega\rangle} = \mathcal{H}$ implies $|\Phi\rangle$ is cyclic. Since the same argument holds for \mathcal{A}' , then $|\Phi\rangle$ is separating for \mathcal{A} . For any $A \in \mathcal{A}$, we have that $(US_{\Omega}U'^*)A|\Phi\rangle = US_{\Omega}U'^*AU'U|\Phi\rangle = U_R S_{\Omega}(AU)|\Omega\rangle = U(AU)^*|\Omega\rangle = A^*|\Omega\rangle \Rightarrow S_{\Omega, \Phi} = US_{\Omega}U'^*$. \square

Corollary 5.9. *In the context of the above lemma, if $|\Omega\rangle$ and $|\Phi\rangle$ are vector representatives of the states ω and ϕ , then $S(\phi | \omega) = \langle \Phi | U'K_{\Omega}U'^* | \Phi \rangle = \langle \Omega | U^*K_{\Omega}U | \Omega \rangle$.*

The above corollary shows explicitly that the RE does not depend on the unitary U' . This is expected because the RE is a measure of distinguishability of the states in \mathcal{A} , and hence, it has to be invariant under unitary changes of the states outside \mathcal{A} .

Now we apply the corollary 5.9 to the case of a coherent state, i.e. $U = W(f_R)$ with $\text{supp}(f_R) \subset \mathcal{W}_R$. According to Bisognano-Wichmann theorem (theorem 4.5), we know that the modular Hamiltonian of the vacuum vector $|0\rangle$ is given by $K_0 = 2\pi K_1$, where K_1 is the infinitesimal boost operator. Remembering that the second quantized Poincaré operator $U(\Lambda_1^s, 0) = e^{iK_1 s}$, acting on the Fock space \mathcal{H}_0 , is constructed from the Poincaré unitary operator $u(\Lambda_1^s, 0) = e^{ik_1 s}$, acting on the one-particle Hilbert space \mathfrak{H} , then we have that

$$\begin{aligned} S(\omega_f | \omega_0) &= \langle 0 | U^* K_0 U | 0 \rangle = 2\pi \langle 0 | W(f_R)^* K_1 W(f_R) | 0 \rangle = 2\pi \langle e^{if_R} | K_1 | e^{if_R} \rangle \\ &= 2\pi e^{-\|f_R\|_{1p}^2} \sum_{n=0}^{\infty} \frac{(-i)^n (i)^n}{n!} \langle f_R^{\otimes n} | K_1 | f_R^{\otimes n} \rangle \\ &= 2\pi e^{-\|f_R\|_{1p}^2} \sum_{n=0}^{\infty} \frac{n}{n!} \langle f_R^{\otimes n} | (k_1 f_R) \otimes f_R^{\otimes n-1} \rangle \\ &= 2\pi e^{-\|f_R\|_{1p}^2} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \langle f_R | k_1 | f_R \rangle_{1p} \langle f_R | f_R \rangle_{1p}^{n-1} \\ &= 2\pi e^{-\|f_R\|_{1p}^2} \langle f_R | k_1 | f_R \rangle_{1p} e^{\|f_R\|_{1p}^2} = 2\pi \langle f_R | k_1 | f_R \rangle_{1p}. \end{aligned} \quad (5.104)$$

Thus, the RE between the coherent state and the vacuum, can be expressed in the one-particle Hilbert space \mathfrak{H} in terms of the expectation value of the boost operator k_1 in the vector $E(f) \in \mathfrak{H}$, which generates the underlying coherent state. In the end, following from (5.104), we get the following theorem.

Theorem 5.10. *Let be $f_L, f_R \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ with $\text{supp}(f_L) \in \mathcal{W}_L$, $\text{supp}(f_R) \in \mathcal{W}_R$, and $f = f_L + f_R$. Then, the relative entropy between the coherent state ω_f and the vacuum ω_0 , for the algebra of the right Rindler wedge \mathcal{W}_R is*

$$S(\omega_f | \omega_0) = 2\pi \int_{x^1 > 0} d^{d-1}x x^1 \frac{1}{2} \left(\left(\frac{\partial F}{\partial x^0} \right)^2 + |\nabla F|^2 + m^2 F^2 \right) \Big|_{x^0=0}, \quad (5.105)$$

where $F(x) = \int_{\mathbb{R}^d} \Delta(x-y) f(y) d^d y = \int_{\mathbb{R}^d} \Delta(x-y) [f_L(y) + f_R(y)] d^d y$. In addition, formula (5.105) does not depend on the chosen function f_L (with support in \mathcal{W}_L).

Proof. A straightforward calculation explained in appendix C.1, allows us to obtain (C.10) from (5.104). However, there are two differences between (C.10) and (5.105) (beyond the obvious 2π in front of the expression). The first one is that, in (C.10) the integral is along the whole space \mathbb{R}^{d-1} , and the second one is that the function F in (C.10) is computed using only f_R . To finally pass from (C.10) to (5.105) we have to do the following two changes. First we notice that $\text{supp}(f_R) \subset \mathcal{W}_R$ implies that $\text{supp}(F|_{x^0=0}) \subset \Sigma_R$, and hence, we can replace the integration region in (C.10) by Σ_R . In the same way, since $\text{supp}(f_L) \subset \mathcal{W}_L$, we have that the function $F_L(x) := \int_{\mathbb{R}^d} \Delta(x-y) f_L(y) d^d y$ vanishes along Σ_R , and hence, (5.105) holds. This also implies that (5.105) does not depend on f_L . \square

As a remark, the outcome of the above theorem coincides with (5.15) for the canonical stress tensor (5.7) as well as with any other improvement term, since the generating function of the coherent state identically vanishes along the entanglement surface $x^1 = 0$.

5.5.2. Hard case: $f \neq f_L + f_R$

In this section, we assume that the function $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ has $\text{supp}(f) \not\subset \mathcal{W}_L, \mathcal{W}_R$. Moreover, we assume that $\text{supp}(f)$ is compact to avoid possible complications coming from integrals along regions of infinite size. In the end, we are interested in the behavior of the RE around the entanglement surface $\partial\Sigma_R = \{\bar{x} \in \mathbb{R}^{d-1} : x^1 = 0\}$, which can be captured with a compactly supported coherent state.

Before we continue, we want to remark that, in this case, the RE must be finite. The proof is as follows. Since $\text{supp}(f)$ is compact, there exists a bigger right wedge $\tilde{\mathcal{W}}_R \supset \mathcal{W}_R$ such that $W(f) \in \tilde{\mathcal{W}}_R$. Then the RE between the coherent and the vacuum in the algebra $\mathcal{A}(\tilde{\mathcal{W}}_R)$ is as the one computed in the previous section, which is finite because the generating

function f is smooth. Then by monotonicity, the RE for the original wedge \mathcal{W}_R must be finite. In particular, we are allowed to use expression (3.52).

The first question which arises is if we could split the unitary into two unitaries, one belonging to the right wedge \mathcal{W}_R and the other to the left wedge \mathcal{W}_L . In other words, if there exist unitaries $U_R \in \mathcal{A}_{\mathcal{W}}$ and $U_L \in \mathcal{A}_{\mathcal{W}'}$ such that $W(f) = U_L U_R$. Unfortunately, the answer is negative for the most general interesting case. This fact arises when we try to explicitly split $W(f)$. To begin with, it seems natural to split the function f simply as

$$f_R(x) := \Theta_{\mathcal{W}_R}(x) f(x), \quad (5.106)$$

$$f_L(x) := \Theta_{\mathcal{W}_L}(x) f(x), \quad (5.107)$$

where $\Theta_{\mathcal{W}_R}$ is the characteristic function of the right Rindler wedge (equivalently for $\Theta_{\mathcal{W}_L}$). However, it will lead to a wrong result, since $f_R + f_L \neq f$. Moreover, if for example we start with a function f supported in the upper light-cone $V^+ := \{x \in \mathbb{R}^d : x^0 > |\bar{x}|\}$, then equation (5.106) implies that $f_R \equiv 0$ and hence we obtain $S(\omega_f | \omega_0) = 0$, which is obviously the wrong result. To make a consistent splitting, we must use the relations explained in subsection 5.3.3. Given the spacetime function $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, we can construct $f_\varphi, f_\pi \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ satisfying the relation (5.90). The correct result is to split the functions f_φ, f_π , which are the initial data at $x^0 = 0$ of the Klein-Gordon solution generated by f . The assumption $\text{supp}(f) \not\subset \mathcal{W}_L, \mathcal{W}_R$ implies that f_φ and f_π do not vanish in any open neighborhood of the entanglement surface $\partial\Sigma_R$. Now, we would like to write

$$f_\varphi = f_{\varphi,L} + f_{\varphi,R} \quad \text{and} \quad f_\pi = f_{\pi,L} + f_{\pi,R}, \quad (5.108)$$

with $\text{supp}(f_{\nu,L}) \in \Sigma'_R$ and $\text{supp}(f_{\nu,R}) \in \Sigma_R$ ($\nu = \varphi, \pi$). The right way to do this is defining

$$f_{\nu,L}(\bar{x}) := f_\nu(\bar{x}) \cdot \Theta(-x^1) \quad \text{and} \quad f_{\nu,R}(\bar{x}) := f_\nu(\bar{x}) \cdot \Theta(x^1), \quad (5.109)$$

where Θ is the usual step Heaviside function. The problem is that $f_{\nu,L}$ and $f_{\nu,R}$ are no longer smooth, and nothing guarantees that $E_\nu(f_{\nu,R}) \in \mathfrak{H}_\nu$ (the same problem occurs for $f_{\nu,L}$). More precisely, since $f_{\nu,R} \in L^2(\mathbb{R}^{d-1}, \mathbb{R}) = H^0(\mathbb{R}^{d-1}, \mathbb{R})$, and because of the inclusions (see appendix B)

$$H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \subset H^0(\mathbb{R}^{d-1}, \mathbb{R}) \subset H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}), \quad (5.110)$$

we have that $f_{\varphi,R} \in \mathfrak{H}_\varphi$ but $f_{\pi,R} \notin \mathfrak{H}_\pi$. In other words, $f_{\pi,R}$ is not an appropriate smear function for the canonical conjugate field $\pi(\bar{x})$. This problem does not appear because the test function is no longer smooth, it just arises because $f_{\pi,R}$ is no longer continuous. On the other hand, if $f_{\pi,R}$ is continuous, this problem can be solved due to the following lemma.

Theorem 5.11. *Let $f \in L^2(\mathbb{R}^n) \cap C^0(\mathbb{R}^n) \cap C_t^1(\mathbb{R}^n)$ and $\partial_j f \in L^2(\mathbb{R}^n)$ for $j = 1, \dots, n$.¹³ Then $f \in H^1(\mathbb{R}^n)$.*

Proof. See appendix B. □

Then, having this in mind, the strategy we adopt below is to make a splitting for another smear function, which, by construction, we know is continuous.

5.5.2.1. A lemma for the relative modular group

In this subsection, we prove a lemma that gives a general expression for the relative modular flow, under the assumption that some operator can be written as a product of two new operators, one belonging to \mathcal{A} and other to \mathcal{A}' . In the following subsection, we prove that this assumption holds for the free real scalar field. For simplicity and due to the symmetry relation (5.98), in the following, we work with the modular operator $\Delta_{\Phi,0}$ instead of $\Delta_{0,\Phi}$.

As a motivation, we remember from section 3.3.1, that contrary to the modular group Δ_{Ω}^{it} and the relative modular group $\Delta_{\Phi,\Omega}^{it}$, the family of operators $u_{\Phi,\Omega}(t) = \Delta_{\Phi,\Omega}^{it} \Delta_{\Omega}^{it}$ belongs to \mathcal{A} for all $t \in \mathbb{R}$ (see lemma 3.28). This suggests that the computation of $u_{\Phi,\Omega}(t)$ could involve the splitting of some test function, which would lead to a well-defined operator. To gain some intuition, using the lemmas 5.2 and 5.8, we know that

$$u_{\Phi,\Omega}(t) = U^* \Delta_{\Omega}^{it} U \Delta_{\Omega}^{-it} \quad \text{whenever } |\Phi\rangle = U'U |\Omega\rangle \text{ with } U \in \mathcal{A}, U' \in \mathcal{A}'. \quad (5.111)$$

This expression motivates the following lemma.

Lemma 5.12. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a vN factor, $|\Omega\rangle$ a standard vector, $U \in \mathcal{B}(\mathcal{H})$ a unitary such that $U^* \mathcal{A} U = \mathcal{A}$, and $|\Phi\rangle = U |\Omega\rangle$. If there exist families of unitaries¹⁴ $V(t) \in \mathcal{A}$ and $V'(t) \in \mathcal{A}'$ such that*

$$\begin{cases} U^* \Delta_{\Omega}^{it} U \Delta_{\Omega}^{-it} = V(t) V'(t), & \forall t \in \mathbb{R}, \\ V(0) = V'(0) = \mathbf{1}. \end{cases} \quad (5.112)$$

Then, there exists a real function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$ such that

$$\Delta_{\Phi,\Omega}^{it} = e^{-i\alpha(t)} V(t) \Delta_{\Omega}^{it}. \quad (5.113)$$

Proof. We first see that $V(t) \Delta_{\Omega}^{it}$ acts as $\Delta_{\Phi,\Omega}^{it}$ on every $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$. Indeed, we

¹³ $C_t^1(\mathbb{R}^n)$ is the set of piecewise differentiable functions. See appendix B for a precise definition.

¹⁴They are not necessarily one-parameter groups for $t \in \mathbb{R}$.

have that

$$\begin{aligned}
\mathcal{A} \ni V(t) \Delta^{it} A \Delta^{-it} V(t)^* &= V(t) V'(t) \Delta_{\Omega}^{it} A \Delta_{\Omega}^{-it} V(t)^* V'(t)^* \\
&= U \Delta_{\Omega}^{it} U^* \Delta_{\Omega}^{-it} \Delta_{\Omega}^{it} A \Delta_{\Omega}^{-it} \Delta_{\Omega}^{it} U \Delta_{\Omega}^{-it} U^* \\
&= U \Delta_{\Omega}^{it} U^* A U \Delta_{\Omega}^{-it} U^* = \Delta_{\Phi}^{it} A \Delta_{\Phi}^{-it} = \Delta_{\Phi, \Omega}^{it} A \Delta_{\Phi, \Omega}^{-it}, \quad (5.114)
\end{aligned}$$

where we have used 2 in lemma 5.2. Similarly, we have that

$$V(t) \underbrace{\Delta_{\Omega}^{it} A' \Delta_{\Omega}^{-it}}_{\in \mathcal{A}_{\mathcal{W}'}} V(t)^* = V(t) V(t)^* \Delta_{\Omega}^{it} A' \Delta_{\Omega}^{-it} = \Delta_{\Omega}^{it} A' \Delta_{\Omega}^{-it} = \Delta_{\Phi, \Omega}^{it} A' \Delta_{\Phi, \Omega}^{-it}. \quad (5.115)$$

Then, for all $B \in \mathcal{A} \cup \mathcal{A}'$ we have that

$$(V(t) \Delta_{\Omega}^{it}) B (V(t) \Delta_{\Omega}^{it})^* = \Delta_{\Phi, \Omega}^{it} B \Delta_{\Phi, \Omega}^{-it} \Rightarrow [B, (V(t) \Delta_{\Omega}^{it})^* \Delta_{\Phi, \Omega}^{it}] = 0, \quad (5.116)$$

and hence $(V(t) \Delta_{\Omega}^{it})^* \Delta_{\Phi, \Omega}^{it}$ belongs to the center $(\mathcal{A} \cup \mathcal{A}')' = \mathcal{A} \cap \mathcal{A}' = \{\lambda \cdot \mathbf{1}\}$, which is trivial since \mathcal{A} is a factor. This means that there exist a function $\lambda : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\Delta_{\Phi, \Omega}^{it} = \lambda(t) V(t) \Delta_{\Omega}^{it}. \quad (5.117)$$

Moreover, evaluating the above expression at $t = 0$ we get that $\lambda(0) = 1$. Finally, since all operators in (5.117) are unitaries, we must have that $\lambda(t) = e^{-i\alpha(t)}$ for some function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ with $\alpha(0) = 0$, and then, (5.113) holds. \square

Under the hypothesis of the above lemma, we can obtain the relative modular Hamiltonian deriving (5.113) at $t = 0$

$$K_{\Phi, \Omega} = i \left. \frac{d}{dt} \right|_{t=0} \Delta_{\Phi, \Omega}^{it} = i \left. \frac{d}{dt} \right|_{t=0} e^{-i\alpha(t)} V(t) \Delta_{\Omega}^{it} = \alpha'(0) \mathbf{1} + i\dot{V}(0) + K_{\Omega}, \quad (5.118)$$

where the derivative in (5.118) has to be understood as a limit in the strong operator topology of \mathcal{H} [51]. This formula gives a well-defined expression for the relative modular Hamiltonian up to a constant. One way to determine such a constant is using that $\Delta_{\Phi, \Omega}^{it}$ is a one-parameter group of unitaries and must fulfill the concatenation equation

$$\Delta_{\Phi, \Omega}^{it_1} \Delta_{\Phi, \Omega}^{it_2} = \Delta_{\Phi, \Omega}^{i(t_1+t_2)}, \quad \forall t_1, t_2 \in \mathbb{R}. \quad (5.119)$$

We will discuss the computation to determine $\alpha'(0)$ in subsection 5.5.2.3.

5.5.2.2. Relative modular group

In this subsection, we show that lemma 5.12 applies for the free real scalar field and coherent states. More concretely, we have the following theorem.

Theorem 5.13. *For the algebra of the Rindler wedge $\mathcal{A}_{\mathcal{W}}$, a Weyl unitary $U = W(f)$ with $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, the vacuum vector $|0\rangle$ and the vector $|\Phi\rangle = U|0\rangle$, the hypothesis of lemma 5.12 holds. In particular, we have that*

$$\Delta_{\Phi,0}^{it} = e^{i\alpha(s)} W_{\varphi}(g_{\varphi,R}^s) W_{\pi}(g_{\pi,R}^s) \Delta_0^{it} = e^{i\alpha(s)} e^{i\varphi(g_{\varphi,R}^s)} e^{i\pi(g_{\pi,R}^s)} e^{isK_1}, \quad (5.120)$$

where we have denoted $s := -2\pi t$ and

$$g_{\varphi,R}^s(\bar{x}) = -\frac{\partial G^s}{\partial x^0}(0, \bar{x}) \Theta(x^1), \quad (5.121)$$

$$g_{\pi,R}^s(\bar{x}) = G^s(0, \bar{x}) \Theta(x^1), \quad (5.122)$$

$$G^s(x) = \int_{\mathbb{R}^d} \Delta(x-y) [f(\Lambda_1^{-s}y) - f(y)] d^d y. \quad (5.123)$$

Proof. According to the discussion in subsection 5.3.1.3, we have that $\mathcal{A}_{\mathcal{W}}$ is a vN factor. From the Reeh-Schlieder theorem, we have that the vacuum vector $|0\rangle$ is standard. Also, we have that any Weyl unitary satisfies $W(\mathbf{h})^* \mathcal{A}_{\mathcal{W}} W(\mathbf{h}) = \mathcal{A}_{\mathcal{W}}$. From now on, we set $s = -2\pi t$. Replacing $U = W(f)$ into (5.112), we get

$$\begin{aligned} W(f)^* \Delta_0^{it} W(-f) \Delta_0^{-it} &= W(-f) e^{isK_1} W(f) e^{-isK_1} = W(-f) W\left(f_{(\Lambda_1^s, 0)}\right) \\ &= e^{i\text{Im}\langle f|f^s \rangle_{1\mathbb{P}}} W(f^s - f), \end{aligned} \quad (5.124)$$

where we have defined $f^s := f_{(\Lambda_1^s, 0)}$. Applying the decomposition (5.88) to $g^s := f^s - f$, we have that

$$\begin{aligned} W(f)^* \Delta_0^{it} W(f) \Delta_0^{-it} &= e^{i\text{Im}\langle f|f^s \rangle_{1\mathbb{P}}} W(g^s) \\ &= e^{i\text{Im}\langle f|f^s \rangle_{1\mathbb{P}}} e^{i\text{Im}\langle g_{\varphi}^s|g_{\pi}^s \rangle_{1\mathbb{P}}} W_{\varphi}(g_{\varphi}^s) W_{\pi}(g_{\pi}^s), \end{aligned} \quad (5.125)$$

with

$$g_{\varphi}^s(\bar{x}) := -\frac{\partial G^s}{\partial x^0}(0, \bar{x}) = -\cosh(s) \frac{\partial F}{\partial x^0}(\bar{x}^s) + \sinh(s) \frac{\partial F}{\partial x^1}(\bar{x}^s) + \frac{\partial F}{\partial x^0}(0, \bar{x}), \quad (5.126)$$

$$g_{\pi}^s(\bar{x}) := G^s(0, \bar{x}) = F(\bar{x}^s) - F(0, \bar{x}), \quad (5.127)$$

where $\bar{x}^s := (\Lambda_1^{-s}x)_{x^0=0} = (-x^1 \sinh(s), x^1 \cosh(s), \bar{x}_{\perp})$ and

$$G^s(x) := \int_{\mathbb{R}^d} \Delta(x-y) [f^s(y) - f(y)] d^d y. \quad (5.128)$$

Now, we explicitly split the unitaries $W_\varphi(g_\varphi^s)$ and $W_\pi(g_\pi^s)$ in equation (5.125) defining

$$g_{\varphi,R}^s(\bar{x}) := g_\varphi^s(\bar{x}) \Theta(x^1) \quad \text{and} \quad g_{\varphi,L}^s(\bar{x}) := g_\varphi^s(\bar{x}) \Theta(-x^1), \quad (5.129)$$

$$g_{\pi,R}^s(\bar{x}) := g_\pi^s(\bar{x}) \Theta(x^1) \quad \text{and} \quad g_{\pi,L}^s(\bar{x}) := g_\pi^s(\bar{x}) \Theta(-x^1), \quad (5.130)$$

which evidently implies that $g_{\varphi,L}^s + g_{\varphi,R}^s = g_\varphi^s$ and $g_{\pi,L}^s + g_{\pi,R}^s = g_\pi^s$. Moreover, we have that

$$\left. \begin{array}{l} g_{\varphi,R}^s, g_{\varphi,L}^s \in L^2(\mathbb{R}^{d-1}, \mathbb{R}) \subset H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \\ \text{supp}(g_{\varphi,R}^s) \subset \Sigma_R \text{ and } \text{supp}(g_{\varphi,L}^s) \subset \Sigma'_R \end{array} \right\} \Rightarrow g_{\varphi,R}^s \in K_\varphi(\Sigma_R) \text{ and } g_{\varphi,L}^s \in K_\varphi(\Sigma'_R). \quad (5.131)$$

Furthermore, we have that $g_{\pi,R}^s, g_{\pi,L}^s$ are real-valued functions and they satisfy the hypothesis of lemma (5.11). Then, we have that

$$\left. \begin{array}{l} g_{\pi,R}^s, g_{\pi,L}^s \in H^1(\mathbb{R}^{d-1}, \mathbb{R}) \subset H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \\ \text{supp}(g_{\pi,R}^s) \subset \Sigma_R \text{ and } \text{supp}(g_{\pi,L}^s) \subset \Sigma'_R \end{array} \right\} \Rightarrow g_{\pi,R}^s \in K_\pi(\Sigma_R) \text{ and } g_{\pi,L}^s \in K_\pi(\Sigma'_R), \quad (5.132)$$

which means that the splits (5.129-5.130) work. Coming back to (5.125), we have that

$$\begin{aligned} W(f) \Delta_0^{it} W(f)^* \Delta_0^{-it} &= e^{i\text{Im}\langle f|f^s \rangle_{1\text{P}}} e^{i\text{Im}\langle g_\varphi^s|g_\pi^s \rangle_{1\text{P}}} W_\varphi(g_{\varphi,L}^s + g_{\varphi,R}^s) W_\pi(g_{\pi,L}^s + g_{\pi,R}^s) \\ &= e^{i\text{Im}\langle f|f^s \rangle_{1\text{P}}} e^{i\text{Im}\langle g_\varphi^s|g_\pi^s \rangle_{1\text{P}}} W_\varphi(g_{\varphi,L}^s) W_\varphi(g_{\varphi,R}^s) W_\pi(g_{\pi,L}^s) W_\pi(g_{\pi,R}^s) \\ &= e^{i\text{Im}\langle f|f^s \rangle_{1\text{P}}} e^{i\text{Im}\langle g_\varphi^s|g_\pi^s \rangle_{1\text{P}}} \underbrace{W_\varphi(g_{\varphi,R}^s) W_\pi(g_{\pi,R}^s)}_{\in \mathcal{A}_{\mathcal{W}}} \underbrace{W_\varphi(g_{\varphi,L}^s) W_\pi(g_{\pi,L}^s)}_{\in \mathcal{A}'_{\mathcal{W}}}. \end{aligned} \quad (5.133)$$

Finally, replacing $V(t) = W_\varphi(g_{\varphi,R}^s) W_\pi(g_{\pi,R}^s)$ into (5.113), we arrive to (5.120). \square

Remark 5.14. Using that the relative modular flow $\Delta_{\Phi,0}^{it}$ is strongly continuous and that the relative entropy $S(\omega_f | \omega_0)$ is finite (see discussion at the beginning of section 5.5.2), and hence the expression (3.52) holds, it is not difficult to prove that the function $t \mapsto \langle 0 | \Delta_{\Phi,0}^{it} | 0 \rangle$ is continuously differentiable. Furthermore, taking the vacuum expectation value on the r.h.s of (5.120), it can be shown that the function $\alpha(s) \in C^1(\mathbb{R})$.

To conclude, according to (5.118), we have the following expression for the relative modular Hamiltonian

$$K_{\Phi,0} = 2\pi(\alpha'(0) \mathbf{1} + \varphi(h_{\varphi,R}) + \pi(h_{\pi,R}) + K_1), \quad (5.134)$$

where

$$h_{\varphi,R}(\bar{x}) := \frac{d}{ds} \Big|_{s=0} g_{\varphi,R}^s(\bar{x}) = \left(x^1 \frac{\partial^2 F}{(\partial x^0)^2}(0, \bar{x}) + \frac{\partial F}{\partial x^1}(0, \bar{x}) \right) \cdot \Theta(x^1), \quad (5.135)$$

$$h_{\pi,R}(\bar{x}) := \frac{d}{ds} \Big|_{s=0} g_{\pi,R}^s(\bar{x}) = \left(-x^1 \frac{\partial F}{\partial x^0}(0, \bar{x}) \right) \cdot \Theta(x^1). \quad (5.136)$$

Using similar arguments as above, we have that $h_{\varphi,R} \in K_{\varphi}(\Sigma_R)$ and $h_{\pi,R} \in K_{\pi}(\Sigma_R)$.¹⁵

Before we determine the constant $\alpha'(0)$, we want to emphasize its importance

$$S(\omega_f | \omega_0) = \langle 0 | K_{\Phi,0} | 0 \rangle = 2\pi\alpha'(0). \quad (5.137)$$

Thus, the constant $\alpha'(0)$ is, in fact, the desired result for the RE. Regardless of the problem of computing the value of $\alpha'(0)$, expressions (5.134-5.136) give us an explicit exact expression for the relative modular Hamiltonian $K_{\Phi,0}$ up to a constant. It is interesting to notice that the difference $K_{\Phi,0} - K_0$ is just a linear term on the field operators plus a constant term. We expect that this structure holds not just for the Rindler wedge, but for any kind of region as long as $|\Phi\rangle = W(f)|0\rangle$ is a coherent vector.

5.5.2.3. Relative entropy

As we have explained in equation (5.137), we need to determine the constant $\alpha'(0)$ in order to obtain the final result for the RE. Most of the calculations along this section are straightforward, therefore, we present the detailed computations in appendix C.2. As in theorem 5.13, throughout this section we set $s := -2\pi t$.

We start taking the vacuum expectation value on both sides in expression (5.119),

$$\langle 0 | \Delta_{\Phi,0}^{it_1} \Delta_{\Phi,0}^{it_2} | 0 \rangle = \langle 0 | \Delta_{\Phi,0}^{i(t_1+t_2)} | 0 \rangle, \quad (5.138)$$

and we replace the expression (5.120) obtained for the relative modular flow (see equations (C.11-C.12)). Applying $\frac{d}{ds_1} \Big|_{s_1=0} = -\frac{1}{2\pi} \frac{d}{dt_1} \Big|_{t_1=0}$ on both sides in (5.138) (equations (C.13-C.14)) and matching real and imaginary parts separately, we get^{16,17}

$$\alpha'(s_2) - \frac{d}{ds_2} \text{Im} \langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1P} = \alpha'(0) - \frac{d}{ds_1} \Big|_{s_1=0} \text{Im} \langle g_R^{s_1} | g_R^{s_2} \rangle_{1P}, \quad (5.139)$$

$$\frac{d}{ds_1} \Big|_{s_1=0} \|g_R^{s_1+s_2}\|_{1P}^2 = \frac{d}{ds_1} \Big|_{s_1=0} \|g_R^{s_1} + u(\Lambda_1^{s_1}, 0) g_R^{s_2}\|_{1P}^2, \quad (5.140)$$

where $g_R^s := E_{\varphi}(g_{\varphi,R}^s) + E_{\pi}(g_{\pi,R}^s)$. The second equation is useless to determine $\alpha'(0)$, then we focus in the first one, which is a differential equation for $\alpha'(s)$ with the particularity that $\alpha'(0)$ appears on it. To solve it, let us analyze the second term on the right-hand side

¹⁵An explicit computation of the strong derivative in equation (5.134) shows that the vacuum vector $|0\rangle$, any coherent vector, and any vector of finite number of particles belong to the domain of $K_{\Phi,0}$.

¹⁶Analytic properties of the relative modular flow ensures that both sides of (5.138) are continuous differentiable functions on t_1 and t_2 .

¹⁷The $\frac{d}{ds_2}$ in (5.139) appears because in some terms the dependance on s_1 is through $s_1 + s_2$.

in equation (5.139). In appendix C.2, we compute

$$2\text{Im}\langle g_R^{s_1} | g_R^{s_2} \rangle_{1\text{p}} = P(s_1) + Q(s_1, s_2) - R(s_1, s_2) + \gamma(s_2) , \quad (5.141)$$

where

$$P(s_1) := \int_{x^1 > 0} f_\varphi(\bar{x}) f_\pi^{s_1}(\bar{x}) d^{d-1}x - \int_{x^1 > 0} f_\varphi^{s_1}(\bar{x}) f_\pi(\bar{x}) d^{d-1}x , \quad (5.142)$$

$$Q(s_1, s_2) := \int_{x^1 > 0} f_\varphi^{s_1}(\bar{x}) f_\pi^{s_2}(\bar{x}) d^{d-1}x , \quad (5.143)$$

$$R(s_1, s_2) := \int_{x^1 > 0} f_\varphi^{s_2}(\bar{x}) f_\pi^{s_1}(\bar{x}) d^{d-1}x . \quad (5.144)$$

The function γ in (5.141) includes all the s_1 -independent terms, which do not contribute to (5.139). In the same appendix we analyze P , Q , R carefully, and we obtain

$$\left. \frac{dP}{ds_1} \right|_{s_1=0} = \int_{x^1 > 0} d^{d-1}x x^1 \left(\left(\frac{\partial F}{\partial x^0} \right)^2 + (\nabla F)^2 + m^2 F^2 \right) \Big|_{x^0=0} =: \mathbf{S} , \quad (5.145)$$

and

$$\left. \frac{d}{ds_1} (Q - R) \right|_{s_1=0} = - \left. \frac{d}{ds_2} (Q - R) \right|_{s_1=0} . \quad (5.146)$$

Turning back to (5.139), we can write

$$\begin{aligned} \alpha'(s_2) - \left. \frac{d}{ds_2} \text{Im}\langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1\text{p}} \right|_{s_2=0} &= \alpha'(0) - \left. \frac{d}{ds_1} \right|_{s_1=0} \text{Im}\langle g_R^{s_1} | g_R^{s_2} \rangle_{1\text{p}} \\ &= \alpha'(0) - \frac{1}{2} \left. \frac{d}{ds_1} \right|_{s_1=0} (P(s_1) + Q(s_1, s_2) - R(s_1, s_2)) \\ &= \alpha'(0) - \frac{1}{2} \mathbf{S} + \frac{1}{2} \left. \frac{d}{ds_2} (Q(0, s_2) - R(0, s_2)) \right|_{s_2=0} . \end{aligned} \quad (5.147)$$

Then, integrating this last equation respect to s_2 , we arrive to

$$\alpha(s_2) - \text{Im}\langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1\text{p}} = \alpha'(0) s_2 - \frac{1}{2} \mathbf{S} s_2 + \frac{1}{2} (Q(0, s_2) - R(0, s_2)) , \quad (5.148)$$

where we have used that $g_{\varphi,R}^{s_2=0} = g_{\pi,R}^{s_2=0} = 0 \Rightarrow \text{Im}\langle g_{\varphi,R}^{s_2=0} | g_{\pi,R}^{s_2=0} \rangle_{1\text{p}} = 0$, and $Q(0, 0) - R(0, 0) = 0$, which follows from the definitions of Q and R . To determine $\alpha'(0)$, we invoke the KMS-condition stated in theorem 3.30. Setting $A = B = \mathbf{1}$ in equation (3.50), and simply calling $G(z)$ to the underlying function, we have that

$$G(t) = \langle 0 | \Delta_{\Phi,0}^{it} | 0 \rangle \xrightarrow{t \rightarrow -i} G(-i) = \langle \Phi | \Phi \rangle = 1 . \quad (5.149)$$

In terms of the real variable $s = -2\pi t$, the function $G(s)$ is analytic on $\mathbb{R} + i(0, 2\pi)$, and relation (5.149) must hold for $s \rightarrow 2\pi i$. Using (5.120), we have that

$$G(s) = e^{i\alpha(s)} \langle 0 | e^{i\varphi(g_{\varphi,R}^s)} e^{i\pi(g_{\pi,R}^s)} | 0 \rangle = e^{i\alpha(s) - i\text{Im}\langle g_{\varphi,R}^s | g_{\pi,R}^s \rangle_{1\text{P}} - \frac{1}{2} \|g_R^s\|_{1\text{P}}^2}, \quad (5.150)$$

and hence

$$i\alpha(s) - i\text{Im}\langle g_{\varphi,R}^s | g_{\pi,R}^s \rangle_{1\text{P}} - \frac{1}{2} \|g_R^s\|_{1\text{P}}^2 \xrightarrow{s \rightarrow 2\pi i} i2n\pi, \quad n \in \mathbb{Z}. \quad (5.151)$$

Taking this into account, we come back to (5.148) and write

$$i\alpha(s) - i\text{Im}\langle g_{\varphi,R}^s | g_{\pi,R}^s \rangle_{1\text{P}} - \frac{1}{2} \|g_R^s\|_{1\text{P}}^2 = i\alpha'(0)s - \frac{i}{2} \mathbf{S}s + \frac{i}{2} (Q(0, s) - R(0, s)) - \frac{1}{2} \|g_R^s\|_{1\text{P}}^2. \quad (5.152)$$

Before we take the limit $s \rightarrow 2\pi i$ in the above expression, we may notice that $\bar{x}^s = (-x^1 \sinh(s), x^1 \cosh(s), \bar{x}_\perp) \xrightarrow{s \rightarrow 2\pi i} (0, \bar{x})$, which informally suggests that

$$g_R^s \xrightarrow{s \rightarrow 2\pi i} 0 \implies \|g_R^s\|_{1\text{P}}^2 \xrightarrow{s \rightarrow 2\pi i} 0, \quad (5.153)$$

$$f_\nu^s \xrightarrow{s \rightarrow 2\pi i} f_\nu \implies Q(0, s) - R(0, s) \xrightarrow{s \rightarrow 2\pi i} 0, \quad \text{where } \nu = \varphi, \pi. \quad (5.154)$$

More appropriately, in appendix C.3, we prove that the function

$$N(s) := \frac{i}{2} (Q(0, s) - R(0, s)) - \frac{1}{2} \|g_R^s\|_{1\text{P}}^2, \quad s \in \mathbb{R}, \quad (5.155)$$

can be analytically continued on the strip $\mathbb{R} + i(0, 2\pi)$ and that $\lim_{s \rightarrow 2\pi i} N(s) = 0$. Then, taking the limit $s \rightarrow 2\pi i$ in (5.152), we obtain

$$i2n\pi = -\alpha'(0)2\pi + \frac{1}{2} \mathbf{S}2\pi. \quad (5.156)$$

Since $\alpha'(0), \mathbf{S} \in \mathbb{R}$, then, it must be that $n = 0$. Thereby, we finally get $\alpha'(0) = \frac{1}{2} \mathbf{S}$.

These computation can be summarized in the following theorem, which generalizes the theorem 5.10.

Theorem 5.15. *Let $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ with $\text{supp}(f)$ compact. Then, the relative entropy between the coherent state ω_f and the vacuum state ω_0 for the algebra of the right Rindler wedge \mathcal{W}_R is*

$$S(\omega_f | \omega_0) = 2\pi \int_{x^1 > 0} d^{d-1}x x^1 \frac{1}{2} \left(\left(\frac{\partial F}{\partial x^0} \right)^2 + |\nabla F|^2 + m^2 F^2 \right) \Big|_{x^0=0}, \quad (5.157)$$

where $F(x) = \int_{\mathbb{R}^d} \Delta(x-y) f(y) d^d y$. In addition, formula (5.157) only depends on the behavior of f in $\mathbb{R}^d - \mathcal{W}'_R$.

In conclusion, we have shown that formula (5.15) holds for the canonical stress-tensor (5.7).

5.6. Conclusions of the chapter

In this chapter, we proposed to compute the RE between a coherent state and the vacuum for the algebra of the Rindler wedge in the theory of a free real scalar field. We adopted a complete algebraic perspective, using Araki formula for RE and all the tools of modular theory introduced in section 3.3.1. The motivation to do that has two faces. On one hand, there was not in the literature many examples of mathematically rigorous computations of the RE using techniques of AQFT. It was very instructive for us to understand the way that all the theoretical material about modular theory of vN algebras is realized in a concrete model of QFT. On the other hand, as we have discussed at the beginning of this chapter, the use of non-rigorous methods leads to an ambiguous formula for the RE. Such ambiguities, which come from possible improving terms in the energy stress tensor, can be captured with the help of coherent states.

When the support of the coherent state is contained inside the Rindler wedge, the computation of the RE is simple as we have explained in section 5.5. This result was already known [118]. However, it is useless to solve the mentioned ambiguity since, in that case, the boundary term, in the heuristic formula of the RE, trivially vanishes independently of the chosen improving for the stress tensor.

On the other hand, the more interesting case, when the coherent state has non-vanishing density along the entanglement surface, is much harder to compute. To begin with, we showed that the result of the previous easy case does not apply here, and hence, we must use more sophisticated tools. The conclusion was that the operator $u_{\Phi,\Omega}(t) := \Delta_{\Phi,0}^{it} \Delta_0^{-it}$, which we have introduced in section 3.3.1, is easier to be computed than the modular group $\Delta_{\Phi,0}^{it}$ itself. The main reason is that the group $u_{\Phi,\Omega}(t) \in \mathcal{A}$, even though the modular group does not. In this way, we determined unambiguously the group $u_{\Phi,0}(t)$ imposing all its defining properties (see lemma 3.28). In particular, we showed that our candidate $u_{\Phi,0}(t)$ satisfies the KMS-condition. In the end, the expression (5.15) for the RE holds for the canonical stress tensor. This was already expected since, as we have shown in section 5.2, the appearance of any non-trivial improving term to the canonical stress tensor leads to an expression for the RE that does not satisfy positivity nor monotonicity.

Chapter 6

Entropy and modular Hamiltonian for the chiral current

There has been much progress in understanding local statistical properties of the vacuum reduced to a region in a general $d = 2$ CFT. The Rényi entropies for integer index n have been explicitly computed for several models of interest [119–124]. However, analytic continuation to $n \rightarrow 1$ to obtain the entanglement entropy has shown to be a difficult task. The computation of the vacuum modular Hamiltonians for two intervals has also proved elusive [125].

In this chapter, we calculate the analytic form of the vacuum modular Hamiltonian for a two-interval region and the algebra of the free chiral current. This theory may be identified with the chiral derivatives $J(x) = \partial\phi(x)$ of the free massless scalar field $\phi(x)$ in $d = 2$. We also explicitly compute the MI between these intervals. This model shows a failure of the duality condition (assumption 4 in definition 2.75) for the algebras assigned to two intervals. This fact is translated into a loss of a symmetry property for the MI usually associated with modular invariance [126]. Contrary to the case of the free massless fermion, the modular Hamiltonian turns out to be completely non-local. The calculation is done by diagonalizing the reduced density matrix by computing the eigensystem of a correlator kernel operator, as we have explained in section 4.3. The eigenvectors are obtained by a novel method that involves solving an equivalent problem for a holomorphic function in the complex plane, where multiplicative boundary conditions are imposed on the intervals. Using the same technique, we also rederive the free chiral fermion modular Hamiltonian in a more transparent way than the one used in [36].

We start in section 6.1 giving a brief summary of chiral CFTs. Among all the chiral CFTs, we remark the two models which we study along this chapter: the free fermion and the free current. In section 6.2, we explain in detail how to compute the eigenvectors of the correlator kernel with the novel method mentioned above. In this way, we rederive the

modular Hamiltonian for the n -interval case. There we also introduce the main ideas to be used, in the more complicated case, of the chiral current. However, the treatment of the scalar case is self-contained and a reader not interested in the discussion of the fermion field can start directly in section 6.3. There we describe the algebra of the free chiral current and the relevant kernels. In subsection 6.3.1, we show, for the case of a single interval, the diagonalization of the correlator kernel, and we explicitly compute the entropy and the modular Hamiltonian. The case of two intervals is dealt with in subsection 6.3.2. The treatment is similar to the one of a single interval, but with some extra technical difficulties. We compare the results with the numerical calculations done in subsection 6.3.3. Finally, in section 6.4, we end with the conclusions.

6.1. Chirals CFTs

It is a very well-know fact that the stress-energy tensor of a CFT in $1 + 1$ dimensions decouples into chiralities [127]

$$T(x^0, x^1) = T_+(x^0 + x^1) + T_-(x^0 - x^1) . \quad (6.1)$$

Each component T_{\pm} of the stress tensor could be thought of as living in a one-dimensional QFT over the null line $x_{\pm} := x^0 \pm x^1$. In fact, this behavior is quite general and allows us to consider one-dimensional CFTs, called chiral CFTs, as primary objects, and then construct any $d = 2$ CFT as direct sum of tensor products of chiral CFTs (see [128] and discussion around equation (6.7)). Roughly speaking, a chiral CFT is an AQFT as the one described in chapter 2, but with two main differences:

1. The spacetime is the null-line, which is identified with \mathbb{R} , and the local algebras are attached to open regions $A \subset \mathbb{R}$.
2. The theory is covariant under the Möbius group, which is formed by the global conformal transformations on the null-line.

A chiral CFT could be alternatively described along the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, which is the one-point compactification of the null-line. The map which transforms bijectively the unit circle S^1 onto the null line (+ the point of infinity) is called *Cayley transform*

$$z \in S^1 \mapsto C(z) := i \frac{1-z}{1+z} \in \mathbb{R} \cup \{\infty\} . \quad (6.2)$$

According to this map, any quantum field or local algebra in the circle S^1 can be unambiguously mapped to a quantum field or a local algebra in the line. This alternative is very useful to perform calculations.

The Möbius group Möb could be defined equivalently as:

- a. The set of all transformations on the circle S^1 of the form

$$z \mapsto \frac{\alpha z + \beta}{\beta^* z + \alpha^*}, \quad (6.3)$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$. In this way, we see that $\text{Möb} \cong SU(1, 1) / \{\mathbf{1}, -\mathbf{1}\}$.

- b. The set of all transformation of the null-line \mathbb{R} of the form

$$x \mapsto \frac{ax + b}{cx + d}, \quad (6.4)$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. In this way, we see that $\text{Möb} \cong PSL(2, \mathbb{R})$.

Of course, both definitions are mapped one into the other by means of the Cayley transformation (6.2). The Möbius group is generated by three one-parameter subgroups: R_ϕ ($\phi \in \mathbb{R}$) corresponding to rotations on the unit circle, and T_s ($s \in \mathbb{R}$) and D_λ ($\lambda > 0$) corresponding respectively to translations and dilations on the null-line. For any $g \in \text{Möb}$ and any open region $A \subset \mathbb{R}$, we denote by gA the transformed open set accordingly to (6.4). For completeness, we give the definition of a chiral CFT.

Definition 6.1. An *algebraic chiral CFT* (in the vacuum representation) is an assignment to every open set $A \subset \mathbb{R}$ a vN subalgebra $\mathcal{A}(A) \subset \mathcal{B}(\mathcal{H}_0)$, i.e.

$$A \subset \mathbb{R} \mapsto \mathcal{A}(A) \subset \mathcal{B}(\mathcal{H}_0), \quad (6.5)$$

satisfying the following axioms.

1. (*generating property*) $\mathcal{B}(\mathcal{H}_0) = \bigvee_{A \subset \mathbb{R}} \mathcal{A}(A)$.
2. (*isotony*) For any open sets $A_1 \subset A_2$, then $\mathcal{A}(A_1) \subset \mathcal{A}(A_2)$.
3. (*locality*) For any open set A , then $\mathcal{A}(A') \subset \mathcal{A}(A)'$, where $A' := \mathbb{R} - \overline{A}$.¹
4. (*Möbius covariance*) There is a unitary (strong continuous) representation $g \in \text{Möb} \mapsto U(g) \in \mathcal{B}(\mathcal{H}_0)$, such that $\mathcal{A}(gA) = U(g) \mathcal{A}(A) U(g)^*$ for all $A \subset \mathbb{R}$ and all $g \in \text{Möb}$.
5. (*positive energy*) The self-adjoint generator H of the one-parameter translation subgroup $s \in \mathbb{R} \mapsto U(T_s)$ is a positive operator.² H is called the *chiral Hamiltonian*.

¹In the case of fermions nets, we have to relax this axiom imposing twisted locality respect to a \mathbb{Z}_2 -grading (see section 2.2.4).

²An equivalent requirement is that the self-adjoint generator of the rotation subgroup $\phi \in \mathbb{R} \mapsto U(R_\phi)$ is positive [128].

6. (*vacuum*) There exists a unique (up to a phase) Möb-invariant vector $|0\rangle \in \mathcal{H}_0$, which is called the vacuum vector.

Remark 6.2. It could be given an equivalent definition in terms of a net of vN algebras on the unit circle S^1 .

The construction of a two-dimensional chiral CFT in Minkowski space using two chiral CFTs is as follows. Let us take the basis $\{e_+, e_-\} \subset \mathbb{R}^2$, where $e_\pm := e^0 \pm e^1$, and define the null rays $\mathcal{N}_\pm := \{\lambda e_\pm : \lambda \in \mathbb{R}\}$. Then, each point $x \in \mathbb{R}^2$ can be uniquely decomposed as $x = x_+ e_+ + x_- e_-$, and in particular, each double cone $W \subset \mathbb{R}^2$ can be parametrized by two one-dimensional intervals $A_\pm := (a_\pm, b_\pm) \in \mathcal{N}_\pm$ (see figure 6.1). Then, given two chiral CFTs

$$A_\pm \subset \mathbb{R} \mapsto \mathcal{A}_\pm(A_\pm) \subset \mathcal{B}(\mathcal{H}_\pm), \quad (6.6)$$

we can define the net of vN algebras

$$W \text{ (double cone)} \mapsto \mathcal{A}_{2d}(W) := \mathcal{A}_+(A_+) \otimes \mathcal{A}_-(A_-) \subset \mathcal{B}(\mathcal{H}_+) \otimes \mathcal{B}(\mathcal{H}_-). \quad (6.7)$$

Such a net satisfies all the axioms of definitions 2.56 and 2.58. Furthermore, it is covariant with respect to two copies of the Möbius group $\text{Möb} \times \text{Möb} \supsetneq \mathcal{P}_+^\uparrow$. In particular, the vacuum vector of the $d = 2$ CFT is just the tensor product of the vacuum vectors of each chiral CFT, i.e. $|0\rangle = |0\rangle_+ \otimes |0\rangle_-$. General causally complete regions are handled in a similar fashion.

It is important to remark that the tensor product structure (6.7) is not the most general possibility. In fact, in a generally non-chiral $d = 2$ CFT, the algebra of a causally complete region $\mathcal{A}_{2d}(\mathcal{O})$ may contain further operators than the ones belonging $\mathcal{A}_+(A_+) \otimes \mathcal{A}_-(A_-)$. At the end of this chapter, in section 6.3.2.6, we show how these features appear in the case of the free massless scalar field.³

Whatever the case, here we are interested in one chiral CFT. Once we have identified the chiral CFTs as a special kind of AQFTs, we could pursue the study of the entanglement in such models. In particular, here we study the modular Hamiltonian and the EE for two specific models: the free chiral fermion (section 6.2) and the free chiral current (section 6.3).

³Moreover, in the most general case, the Hilbert space of a two-dimensional CFT could be defined as a direct sum of tensor products of the form

$$\mathcal{H} := \bigoplus_{k=0}^K \mathcal{H}_{+,k} \otimes \mathcal{H}_{-,k}, \quad (6.8)$$

where $K \in \mathbb{N}_0 \cup \{\infty\}$ and the Hilbert spaces $\mathcal{H}_{\pm,k}$ carry unitaries highest weight representations of Virasoro algebra $(c_{\pm,k}, h_{\pm,k})$, where $c_{\pm,k}$ are the corresponding central charges. Local algebras for two-dimensional causally complete regions could be defined in similar fashion as in (6.7), but regarding the richer structure coming from (6.8) (see [128]).

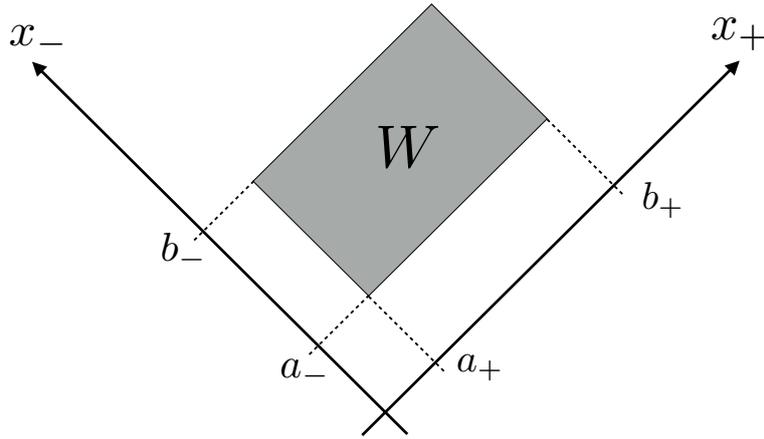


Figure 6.1: The double cone $W \subset \mathbb{R}^2$ can be parametrized by two one-dimensional intervals $A_{\pm} = (a_{\pm}, b_{\pm}) \in \mathcal{N}_{\pm}$.

6.2. The free chiral fermion

The free chiral fermion is usually defined as a quantum field $\psi(x)$ satisfying the anticommutation relations $\{\psi(x), \psi^{\dagger}(y)\} = \delta(x - y)$. The vacuum state is a Gaussian state with two-point correlator⁴

$$C(x - y) := \langle 0 | \psi(x) \psi^{\dagger}(y) | 0 \rangle = \frac{1}{2} \delta(x - y) + \frac{i}{2\pi} \frac{1}{x - y}, \quad (6.9)$$

where the distribution on the left hand side is understood as the principal value regularization. (6.9) is a projector when it acts, as a kernel, on the full line. On a region A , it is an Hermitian operator with continuous eigenvalues in the range $(0, 1)$ [36]. From the algebraic standpoint, given a test function $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$, the operator

$$\psi(f) := \int_{\mathbb{R}} \psi(x) f(x) dx, \quad (6.10)$$

is a well-defined operator acting on the fermionic Fock Hilbert space \mathcal{H}_0 . This Hilbert space is defined as the anti-symmetric tensor product of the one-particle Hilbert space of massless particles of helicity $\frac{1}{2}$. The one-particle Hilbert space carries a unitary representation of Möb , which naturally extends to a unitary representation of Möb on \mathcal{H}_0 . For any open set $A \subset \mathbb{R}$, the local algebras are defined as

$$\mathcal{A}_f(A) := \{\psi(f) : \text{supp}(f) \subset A\}'' . \quad (6.11)$$

It is not difficult to see that such a collection of algebras satisfies all the axioms of the definition 6.1.

Now we consider a region $A := (a_1, b_1) \cup (a_2, b_2) \cup \dots \cup (a_n, b_n)$ formed by n disjoint

⁴Changing the sign of the imaginary part in expression (6.9) corresponds to changing chirality.

intervals ($a_j < b_j < a_{j+1}$ for all $j = 1, \dots, n$). According to the discussion in section 4.3, to obtain the modular Hamiltonian we need to solve the spectrum of the correlator (6.9) reduced to the region A , i.e. $C_A(x, y) := C(x, y)|_{x, y \in A}$. Then, the modular Hamiltonian is given by the equations (4.58) and (4.61), which means

$$K = \int_{A \times A} dx dy \psi^\dagger(x) \mathcal{K}(x, y) \psi(y), \quad (6.12)$$

where the kernel \mathcal{K} is

$$\mathcal{K} = -\log(C_A^{-1} - 1). \quad (6.13)$$

This last equation is understood as an operator equation, where the action of the operators is defined through their kernels.

A complete description of the reduced density matrix of a massless fermion field for multi-interval regions was given in [36]. This was achieved by diagonalizing the correlator kernel in the region, using previous results in the literature about singular kernels of the Cauchy type [129]. In the following subsection, we make this diagonalization more transparent by mapping the problem of integral equations in one dimension to one of partial differential equations in two dimensions following [38]. The form of the eigenvectors as well as its main properties are easily derived using this trick.

6.2.1. An equivalent problem in the complex plane

In this section, we will relate the original problem of solving the spectrum of C_A as a kernel in A , to a new problem about a function in the complex plane. For such a purpose, we think the n intervals A as included in the real axis of the complex plane. For each $\lambda \in \mathbb{R}$, we consider the following problem about a function $S(z)$ in the complex plane

$$S(z) \text{ analytic in } \mathbb{C} - \bar{A}, \quad (6.14)$$

$$S^+(x_1) = \lim_{x_2 \rightarrow 0^+} S(x_1 + ix_2) = \lambda \lim_{x_2 \rightarrow 0^-} S(x_1 + ix_2) = \lambda S^-(x_1), \quad x_1 \in A, \quad (6.15)$$

$$\lim_{z \rightarrow \infty} |z S(z)| < \infty, \quad (6.16)$$

$$\lim_{z \rightarrow \partial A} l_{z, \partial A} S(z) \rightarrow 0, \quad (6.17)$$

where $l_{z, \partial A}$ is the distance from z to the boundary ∂A (formed by $2n$ disjoint points). Thus, $S(z)$ has a cut over A with multiplicative boundary conditions. Consider now the following complex integral

$$\oint dz_2 \frac{1}{z_2 - z_1} S(z_2), \quad z_1 \in \mathbb{C} - \bar{A}, \quad (6.18)$$

where we choose an integration contour that encircles both A and z_1 in the positive (anti-clockwise) direction. This integral vanishes because of (6.16). Writing it as two separated contributions from the pole at z_1 and the integration around the cut A , we get

$$S(z_1) = \frac{1}{2\pi i} \int_A dy \frac{1}{y - z_1} (S^+(y) - S^-(y)) = \frac{1 - \lambda^{-1}}{2\pi i} \int_A dy \frac{1}{y - z_1} S^+(y), \quad (6.19)$$

where we have used the boundary condition (6.15). We remark there are no contributions from the endpoints of the intervals due to (6.17). This equation gives the value for $S(z)$ on any point $z \in \mathbb{C} - \bar{A}$ from its values at the cut A . Taking the limit $z_1 \rightarrow x \in A$ from above, and using

$$\lim_{y \rightarrow 0^+} \frac{1}{x + iy} = \frac{1}{x} - i\pi\delta(x), \quad (6.20)$$

we get

$$\int_A dy C_A(x - y) S^+(y) = \frac{\lambda}{\lambda - 1} S^+(x), \quad (6.21)$$

which means that the boundary value of $S(z)$ plays the role of an eigenvector with eigenvalue $\lambda(\lambda - 1)^{-1}$ for the correlator kernel on A . Since the spectrum of $C_A(x - y)$ is restricted to $(0, 1)$ (see [105]), we have that $\lambda \in (-\infty, 0)$. For later convenience, we write

$$\lambda := -e^{2\pi s}, \quad s \in \mathbb{R}. \quad (6.22)$$

Conversely, suppose we have a solution $S^+(x)$ of the eq. (6.21) for some $\lambda \in \mathbb{R}_{\leq 0}$ with appropriate boundary conditions on the endpoints of the intervals as in (6.17).⁵ Then, equation (6.19) gives a complex-valued function $S(z)$ satisfying all the properties (6.14-6.17). For the boundary condition (6.15), the function $S(z)$, defined in this way, has the boundary value $S^+(x)$ at the upper side of the cut. For the lower side of the cut, we have to use

$$\lim_{y \rightarrow 0^+} \frac{1}{x - iy} = \frac{1}{x} + i\pi\delta(x) \quad (6.23)$$

instead of (6.20), to get the right value $S^-(x) = -e^{-2\pi s} S^+(x)$.

In conclusion, the solutions of the problem in the complex plane (6.14-6.17) are in one-to-one correspondence with the eigenvectors of the correlator kernel (6.9).

⁵This is precisely the boundary condition of the eigenvectors for the vacuum state [105].

6.2.2. Multiplicity and normalization of the eigenvectors

Because of conditions (6.15) and (6.17), the function $S(z)$ must have the following asymptotic behavior when $z \rightarrow \partial A$,

$$S(z) \sim V_{a_j} (a_j - z)^{-1/2+is}, \quad (6.24)$$

$$S(z) \sim V_{b_j} (z - b_j)^{-1/2-is}, \quad (6.25)$$

where $V_{a_j}, V_{b_j} \in \mathbb{C}$ are constants. The symbol \sim means that the difference between the left and right sides in expressions (6.24) and (6.25) are analytic functions on $\mathbb{C} - \bar{A}$ with finite limit when $z \rightarrow \partial A$. Below, we will show that the constants V_{a_j} and V_{b_j} uniquely determine the solution. In order to see this, for each $s \in \mathbb{R}$, we define the Green function $G(z, w)$ for the problem (6.14-6.17), i.e.

$$G(z, w) \text{ analytic on } w \in \mathbb{C} - \{w \in \bar{A} \text{ or } w = z\}, \quad (6.26)$$

$$G(z, w) \sim (z - w)^{-1} \quad \text{when } w \sim z, \quad (6.27)$$

$$\lim_{x_2 \rightarrow 0^+} G(z, x_1 + ix_2) = -e^{-2\pi s} \lim_{x_2 \rightarrow 0^-} G(z, x_1 + ix_2), \quad x_1 \in A, \quad (6.28)$$

and in addition $G(z, w)$ satisfies the two boundary conditions (6.16) and (6.17) as a function of w for each $z \in \mathbb{C}$ fixed.⁶ For $w \rightarrow \partial A$, then we have an expansions analogous to (6.24),

$$G(z, w) \sim U_{a_j}(z) (a_j - w)^{-1/2-is}, \quad (6.29)$$

$$G(z, w) \sim U_{b_j}(z) (w - b_j)^{-1/2+is}. \quad (6.30)$$

Then, the combination $G(z, w)S(w)$ does not have any jump singularity at A as a function of w . On the other hand, it has only simple poles at ∂A and at z , but it does not have a singularity at infinity. Since the sum of all its residues must vanish, we have that

$$S(z) = \sum_{j=1}^n (V_{a_j} U_{a_j}(z) - V_{b_j} U_{b_j}(z)). \quad (6.31)$$

This shows there are at most $2n$ linearly independent solutions to the problem (6.14-6.17) for fixed s , and they can be viewed simply as elements of \mathbb{C}^{2n} . It also shows that any solution which is bounded on ∂A (i.e. $V_{a_j} = V_{b_j} = 0$) must vanish.

Now, we will show that the degeneracy of the space of solutions for each s fixed is at most n . Let us take two solutions $S_1(z)$ and $S_2(z)$ corresponding to the same value s . The function $\tilde{S}_1(z) = (S_1(z^*))^*$ is a solution with parameter $-s$ instead of s . The function $\tilde{S}_1(z)S_2(z)$ does not have a cut, only poles at ∂A . The sum of residues must vanish and

⁶We explicitly change the sign of s in (6.28) respect to (6.15).

we get

$$\sum_{j=1}^n \left((V_{a_j}^1)^* V_{a_j}^2 - (V_{b_j}^1)^* V_{b_j}^2 \right) = 0, \quad (6.32)$$

where $V_{a_j}^k, V_{b_j}^k$ are the coefficients corresponding to S_k ($k = 1, 2$). This means that any two solutions of (6.14-6.17) for the same s must be orthogonal according to the canonical pseudo-inner product of $\mathbb{C}^{n,n}$, which includes the case when the two solutions are the same. The argument to justify why the space of solutions is at most n is as follows. Suppose that the s -valued subspace of solutions is spanned by $\{S_1, \dots, S_{2n}\}$, where each S_k is of the form (6.31). Then, after a diagonalization procedure,⁷ we can get a new set of solutions $\{\tilde{S}_1, \dots, \tilde{S}_{2n}\}$ which spans the same subspace but with the property that $V_{a_j}^k = 0$ for all $j = 1, \dots, n$ for all $k = n + 1, \dots, 2n$. Automatically, because of (6.32), we must have that $V_{b_j}^k = 0$ for all $j = 1, \dots, n$ and for all $k = n + 1, \dots, 2n$, and hence, $\tilde{S}_{n+1}(z) = \dots = \tilde{S}_{2n}(z) = 0$. In conclusion, the s -valued subspace of solutions has dimension at most n . We will show, in the next subsection, that the dimension is exactly n .

Now we make a final comment about the normalization of the eigenvectors $S^+(x, s)$, where we are writing explicitly the dependence of the eigenvectors through the eigenvalues s . Since any two eigenvectors $S_1^+(x, s)$ and $S_2^+(x, s')$ must be orthogonal for $s \neq s'$, we have⁸

$$\int_A dx (S_1^+(x, s))^* S_2^+(x, s') \propto \delta(s - s'). \quad (6.33)$$

In order to orthonormalize the eigenvectors, we need to figure out the proportionality constant in the above equation. For this, we note the delta function can only come from the integration around the endpoints of the intervals on the scalar product. Using the asymptotic expansion of the functions near the endpoints, we arrive at⁹

$$\int_A dx (S_1^+(x, s))^* S_2^+(x, s') = \pi e^{2\pi s} \delta(s - s') \sum_{j=1}^n \left((V_{a_j}^1)^* V_{a_j}^2 + (V_{b_j}^1)^* V_{b_j}^2 \right). \quad (6.34)$$

Note that the two terms inside the parenthesis on the r.h.s. are equal according to (6.32).

6.2.3. Construction of the eigenvectors

In this subsection, we will explicitly construct the eigenvectors of the correlator $C_A(x - y)$ using the relation developed in subsection 6.2.1. Concretely, we will find the general structure of any solution $S(z)$ of the problem (6.14-6.17), and through them, we will obtain

⁷Equivalent to the Gauss-Jordan algorithm used to diagonalize a finite dimensional matrix.

⁸The function $(S_1^+(x, s))^*$ is the complex conjugate of the boundary value of $S_1^+(x, s)$, which is not the same that the boundary value of $(S_1(z, s))^*$. These two operations do not commute.

⁹More precisely, we should write each eigenvector as $S(z) = \sum_{i=1}^n V_{a_i} (a_i - z)^{-1/2+is} + V_{b_i} (z - b_i)^{-1/2-is} + R(z)$ where the function $R(z)$ has finite limit when $z \rightarrow \partial A$. Then, after replacing on the l.h.s. of (6.33), we get that the delta Dirac contributions are of the form (6.34).

the corresponding eigenvectors. In particular, we will show that all eigenspaces of a given eigenvalue s have dimension n . In this subsection, $s \in \mathbb{R}$ is fixed.

We start defining the complex valued function

$$\tilde{\omega}(z) := \sum_{j=1}^n \log \left(\frac{z - a_j}{z - b_j} \right), \quad (6.35)$$

where \log is the principal branch of the complex logarithm, which has a branch cut for $z \in \mathbb{R}_{\leq 0}$. The function $\tilde{\omega}$ is analytic everywhere on the plane except at \bar{A} , where it has a jump discontinuity of the form

$$\tilde{\omega}^+(x) - \tilde{\omega}^-(x) = -2\pi i, \quad x \in A. \quad (6.36)$$

Therefore, the function

$$e^{(is + \frac{1}{2})\tilde{\omega}(z)}, \quad (6.37)$$

satisfies the conditions (6.14), (6.15) and (6.17), but it does not satisfy (6.16). On the other hand, given any solution $S(z)$ of (6.14-6.17), the function

$$f(z) := S(z) e^{-(is + \frac{1}{2})\tilde{\omega}(z)} \quad (6.38)$$

is analytic on $\mathbb{C} - \bar{A}$, and it is also continuous on A , and hence, it is analytic on $\mathbb{C} - \partial A$.¹⁰ Then, $f(z)$ must be some meromorphic function in the whole complex plane, with poles located at the endpoints of the intervals and also possibly at ∞ .¹¹ Since $S(z)$ satisfies (6.17) and because of

$$\lim_{z \rightarrow x + i0^+} \left| e^{(is + \frac{1}{2})\tilde{\omega}(z)} \right| = e^{\pi s} \prod_{j=1}^n \sqrt{\left| \frac{(x - a_j)}{(x - b_j)} \right|}, \quad x \in A, \quad (6.39)$$

it follows that $f(z)$ must be of the form

$$f(z) := \frac{g(z)}{\prod_{j=1}^n (z - a_j)}, \quad (6.40)$$

where $g(z)$ is an entire analytic function. In order to satisfy the last requirement (6.16) for $S(z)$, we have that $g(z)$ must be a polynomial function of degree at most $n - 1$. Taking all this into account, any solution $S(z)$ for the problem (6.14-6.17) must be of the form

$$S(z) = \frac{\sum_{k=0}^{n-1} c_k z^k}{\prod_{j=1}^n (z - a_j)} e^{(is + \frac{1}{2})\tilde{\omega}(z)}, \quad (6.41)$$

¹⁰By Schwartz reflection principle.

¹¹A further analysis prevents the possibility of having essential singularities at such points.

where $c_k \in \mathbb{C}$ parametrize n linearly independent functions. Conversely, it is easy to see that any complex valued function of the form (6.41) is a solution of the problem (6.14-6.17).

Taking the limit of $z \rightarrow A$ from the upper side of the cut on expression (6.41), we obtain the eigenvectors

$$S^+(x) = -i(-1)^{n-l} e^{\pi s} e^{is\omega(x)} \frac{\sum_{k=0}^{n-1} c_k x^k}{\sqrt{-\prod_{j=1}^n (x - a_j)(x - b_j)}}, \quad \text{for } x \in (a_l, b_l), \quad (6.42)$$

where

$$\omega(x) := \lim_{z \rightarrow x+i0^+} \operatorname{Re} \tilde{\omega}(z) = \log \left(-\frac{\prod_{j=1}^n (x - a_j)}{\prod_{j=1}^n (x - b_j)} \right). \quad (6.43)$$

Therefore, there are exactly n degenerate eigenfunctions for the same s . This space of eigenfunctions coincides with the one obtained in [36].

Scalar product

Due to the degeneracy, we have some arbitrariness for the election of the eigenvectors. Such a freedom is encoded in the polynomial $P(x) := \sum_{k=0}^{n-1} c_k x^k$ of equation (6.41). In subsection 6.2.4, we fix such a freedom in order to get an orthonormal basis of eigenvectors. In order to do that, it is useful to have an expression for the scalar product between two eigenvectors in terms of its corresponding polynomials. In the rest of this subsection, we will obtain such a expression.

In equation (6.34), using that the scalar product of two eigenfunctions is proportional to a delta function $\delta(s - s')$, we obtained this scalar product in terms of the coefficients of the expansion of the eigenvectors near the endpoints of the intervals. Here, we will reobtain this result by explicit integration of the product of eigenfunctions.

First we take two solutions $S_1^+(x, s)$ and $S_2^+(x, s')$ of the form (6.42) corresponding to two polynomials $P_1(x)$ and $P_2(x)$. Then, we compute the scalar product

$$\int_A dx S_1^{+*}(x, s) S_2^+(x, s') = -e^{\pi(s+s')} \int_{-\infty}^{+\infty} d\omega e^{-i(s-s')\omega} \sum_{l=1}^n \frac{1}{\omega'(x_l)} \frac{P_1^*(x_l) P_2(x_l)}{\prod_{j=1}^n (x_l - a_j)(x_l - b_j)}, \quad (6.44)$$

where we have changed the integration variable to ω and the sum in (6.44) runs over the distinct solutions of the equation $\omega(x) = \omega$, which are the n simple roots of the polynomial equation

$$-e^{\omega} \prod_{j=1}^n (x - b_j) = \prod_{j=1}^n (x - a_j). \quad (6.45)$$

In each interval $A_l = (a_l, b_l)$, $\omega(x)$ is a monotone increasing function that goes from $-\infty$ at a_l to $+\infty$ at b_l . This fact implies that there exist n distinct simple roots x_l , each one belonging to a distinct interval A_l . In (6.44), x_l is understood as function of ω , i.e. $x_l(\omega)$.

In order to proceed, we will show that the following function of ω

$$K(\omega) := \sum_{l=1}^n \frac{1}{\omega'(x_l)} \frac{Q_{2n-2}(x_l)}{\prod_{j=1}^n (x_l - a_j)(x_l - b_j)}, \quad (6.46)$$

is a constant, i.e. $K(\omega)$ is independent of ω for any polynomial $Q_{2n-2}(x)$ of degree $2n - 2$. Replacing the following expression for $\omega'(x)$

$$\omega'(x) = \frac{\prod_{i=1}^n (x - b_i) \sum_{k=1}^n \prod_{j \neq k} (x - a_j) - \prod_{i=1}^n (x - a_i) \sum_{k=1}^n \prod_{j \neq k} (x - b_j)}{\prod_{j=1}^n (x - a_j)(x - b_j)}, \quad (6.47)$$

in (6.46), we arrive at

$$K(\omega) = \sum_{l=1}^n \frac{Q_{2n-2}(x_l)}{\prod_{i=1}^n (x_l - b_i) \sum_{k=1}^n \prod_{j \neq k} (x_l - a_j) - \prod_{i=1}^n (x_l - a_i) \sum_{k=1}^n \prod_{j \neq k} (x_l - b_j)}. \quad (6.48)$$

Since $\omega = -\infty$ implies $x_l = a_l$ and $\omega = +\infty$ implies $x_l = b_l$, then we have the following particular limits

$$K(-\infty) = \sum_{l=1}^n \frac{Q_{2n-2}(a_l)}{\prod_{i=1}^n (a_l - b_i) \prod_{j \neq l} (a_l - a_j)}, \quad (6.49)$$

$$K(\infty) = - \sum_{l=1}^n \frac{Q_{2n-2}(b_l)}{\prod_{i=1}^n (b_l - a_i) \prod_{j \neq l} (b_l - b_j)}. \quad (6.50)$$

Now, we will show that $K(\omega) = K(-\infty)$, and hence constant. For this, from equation (6.45), we have the following polynomial identity

$$e^\omega \prod_{j=1}^n (x - b_j) + \prod_{j=1}^n (x - a_j) = (e^\omega + 1) \prod_{l=1}^n (x - x_l). \quad (6.51)$$

Evaluating (6.51) on $x = a_k$ (for some $k = 1, \dots, n$) we get

$$\prod_{j=1}^n (a_k - b_j) = (1 + e^{-\omega}) \prod_{l=1}^n (a_k - x_l), \quad (6.52)$$

and taking the derivative of (6.51) respect to x and evaluating at $x = x_l$ (for some $l = 1, \dots, n$), we have

$$\prod_{i=1}^n (x_l - b_i) \sum_{k=1}^n \prod_{j \neq k} (x_l - a_j) - \prod_{i=1}^n (x_l - a_i) \sum_{k=1}^n \prod_{j \neq k} (x_l - b_j) = -(1 + e^{-\omega}) \prod_{i=1}^n (x_l - a_i) \prod_{j \neq l} (x_l - x_j). \quad (6.53)$$

Then, replacing (6.52) in the denominator of (6.49), we get

$$K(-\infty) = (1 + e^{-\omega})^{-1} \sum_{l=1}^n \frac{Q_{2n-2}(a_l)}{\prod_{i=1}^n (a_l - x_i) \prod_{j \neq l} (a_l - a_j)}, \quad (6.54)$$

and replacing (6.53) in the denominator of (6.48), it follows

$$K(\omega) = -(1 + e^{-\omega})^{-1} \sum_{l=1}^n \frac{Q_{2n-2}(x_l)}{\prod_{i=1}^n (x_l - a_i) \prod_{j \neq l} (x_l - x_j)}. \quad (6.55)$$

Hence, the expected relation $K(\omega) = K(-\infty)$ follows from

$$\begin{aligned} (1 + e^{-\omega}) [K(-\infty) - K(\omega)] &= \sum_{l=1}^n \frac{Q_{2n-2}(x_l)}{\prod_{i=1}^n (x_l - a_i) \prod_{j \neq l} (x_l - x_j)} \\ &\quad + \sum_{l=1}^n \frac{Q_{2n-2}(a_l)}{\prod_{i=1}^n (a_l - x_i) \prod_{j \neq l} (a_l - a_j)} = 0, \end{aligned} \quad (6.56)$$

where the last equality to zero is a general fact valid for any polynomial Q_{2n-2} of degree $2n - 2$: evaluating the polynomial in $2n$ arbitrary points y_1, \dots, y_{2n} there is a linear equation that relates the value on the first $2n - 1$ points to the value on y_{2n} . This equation is

$$\sum_{l=1}^{2n} \frac{Q_{2n-2}(y_l)}{\prod_{j \neq l} (y_l - y_j)} = 0. \quad (6.57)$$

Eq. (6.56) follows by specializing on $y_j = a_j$ for $j = 1, \dots, n$, and $y_j = x_{j-n}$ for $j = n + 1, \dots, 2n$.

Since $K(\omega)$ is constant, we have that $K(-\infty) = K(\infty)$, i.e. expressions (6.49) and (6.50) are the same. This, in fact, coincides with the relation (6.32) for the coefficients of the expansions (6.24) and (6.25) of the solutions (6.41) at the endpoints of the intervals. Reading off these coefficients from the explicit form of the solutions, the relation (6.32) writes

$$\begin{aligned} \sum_{j=1}^n (V_{a_j}^1)^* V_{a_j}^2 &= - \sum_{l=1}^n \frac{P_1(a_l)^* P_2(a_l)}{\prod_{i=1}^n (a_l - b_i) \prod_{j \neq l} (a_l - a_j)} \\ &= \sum_{l=1}^n \frac{P_1(b_l)^* P_2(b_l)}{\prod_{i=1}^n (b_l - a_i) \prod_{j \neq l} (b_l - b_j)} = \sum_{j=1}^n (V_{b_j}^1)^* V_{b_j}^2. \end{aligned} \quad (6.58)$$

Let us turn back to the scalar product (6.44). Since the integrand on the Fourier transform in ω is constant, we get

$$\int_A dx S_1^{+*}(x, s) S_2^+(x, s') = -2\pi e^{2\pi s} \delta(s - s') \sum_{l=1}^n \frac{P_1(a_l)^* P_2(a_l)}{\prod_{i=1}^n (a_l - b_i) \prod_{j \neq l} (a_l - a_j)}, \quad (6.59)$$

which coincides with the equation (6.34) because of (6.58).

6.2.4. A complete set of eigenvectors

In order to construct a basis of eigenvectors for each eigenspace of fixed $s \in \mathbb{R}$, we choose the following subset $\{u_s^k\}_{k=1}^n$ of eigenfunctions

$$u_s^k(x) := \frac{(-1)^{l+1}}{N_k} \frac{\prod_{i \neq k} (x - a_i)}{\sqrt{-\prod_{i=1}^n (x - a_i)(x - b_i)}} e^{is\omega(x)}, \quad x \in (a_l, b_l), \quad (6.60)$$

with the normalization factor¹²

$$N_k := \sqrt{2\pi} \left(\frac{\prod_{i \neq k} (a_i - a_k)}{\prod_{i=1}^n (b_i - a_k)} \right)^{1/2}. \quad (6.62)$$

In the rest of this subsection, we will show that the set $\{u_s^k\}_{k=1}^n$ is orthonormal and complete.

The orthonormality of $\{u_s^k\}_{k=1}^n$ follows immediately from equation (6.59), and hence, we have

$$\int_A dx u_s^{k*}(x) u_{s'}^{k'}(x) = \delta_{k,k'} \delta(s - s'). \quad (6.63)$$

The completeness is quite less obvious. The general argument of section 6.2.2 shows that n is the maximal degeneracy and then any n linearly independent vectors should form a complete basis. This fact can be shown explicitly as follows. Using the eigenfunctions (6.60), we have that

$$\sum_{k=1}^n \int_{-\infty}^{+\infty} ds u_s^k(x) u_s^{k*}(y) = 2\pi \sqrt{\prod_{i=1}^n \frac{(x - a_i)(y - a_i)}{(x - b_i)(y - b_i)}} \left(\sum_{k=1}^n \frac{1}{N_k^2 (x - a_k)(y - a_k)} \right) \sum_{l=1}^n \frac{\delta(x - x_l)}{\omega'(x_l)}, \quad (6.64)$$

where $x_l \equiv x_l(\omega(y)) \in A_l$ are the n roots of the polynomial equation (6.45) for $\omega = \omega(y)$. In particular, when $y \in A_l$ then $x_l \equiv y$. We use the following algebraic relation¹³

¹²Note that the expression apparently differs from eq. (36) of [36]. However, we have

$$\sum_{j=1}^n \frac{\prod_{l \neq k} (b_j - a_l)}{(b_j - a_k) \prod_{l \neq j} (b_j - b_l)} = \frac{\prod_{i \neq k} (a_i - a_k)}{\prod_{i=1}^n (b_i - a_k)}, \quad (6.61)$$

and then, both equations are in agreement.

¹³This relation is valid for any complex numbers $a_1, \dots, a_n, b_1, \dots, b_n, x, y$. It can be proven using the definition (6.62) for the normalization constants N_k and decomposing the rational function at both sides into the poles for the variable x .

$$\begin{aligned} \sum_{k=1}^n \frac{1}{N_k^2(x-a_k)(y-a_k)} &= \frac{\prod_{k=1}^n (x-b_k)(y-a_k) - \prod_{k=1}^n (y-b_k)(x-a_k)}{2\pi(x-y) \prod_{k=1}^n (x-a_k)(y-a_k)} \\ &= \frac{P(x,y)}{2\pi \prod_{k=1}^n (x-a_k)(y-a_k)}, \end{aligned} \quad (6.65)$$

where the function

$$P(x,y) := \frac{\prod_{k=1}^n (x-b_k)(y-a_k) - \prod_{k=1}^n (y-b_k)(x-a_k)}{x-y}, \quad (6.66)$$

is a polynomial in x of degree $n-1$ for each fixed y , and its roots are the points $x = x_l$ except when $x_l = y$. As a consequence, the only delta function which survives in (6.64) is for $x = y$, and hence

$$\sum_{k=1}^n \int_{-\infty}^{+\infty} ds u_s^k(x) u_s^{k*}(y) = -\frac{P(x,x)}{\prod_{k=1}^n (x-a_k)(x-b_k)} \frac{1}{\omega'(x)} \delta(x-y). \quad (6.67)$$

In order to get a better expression for $P(x,x)$, from (6.66) we define a new function

$$Q(x,y) = P(x,y)(x-y) = \prod_{k=1}^n (x-b_k)(y-a_k) - \prod_{k=1}^n (y-b_k)(x-a_k), \quad (6.68)$$

which allows us to compute

$$P(x,x) = \partial_x Q(x,y)|_{y=x} = -\omega'(x) \prod_{k=1}^n (x-a_k)(x-b_k). \quad (6.69)$$

Finally, replacing (6.69) into (6.67) we obtain the completeness relation

$$\sum_{k=1}^n \int_{-\infty}^{+\infty} ds u_s^k(x) u_s^{k*}(y) = \delta(x-y). \quad (6.70)$$

6.2.5. Modular Hamiltonian

In this subsection, we rederive the results of [36] about the modular Hamiltonian using the information about the spectral decomposition of the correlator kernel $C_A(x-y)$ obtained in the previous subsections. The modular flow corresponding to this modular Hamiltonian and the entanglement entropy for several intervals have been computed in [36]. Recently, the modular Hamiltonian has also been derived using Euclidean path integral methods in [130]. In [62], the MI between several intervals have been computed using the Araki formula without using a cutoff to compute the EE.

From (6.21) and (6.22), the correlator kernel writes

$$C_A(x-y) = \sum_{k=1}^n \int_{-\infty}^{+\infty} ds u_s^k(x) \frac{1 + \tanh(\pi s)}{2} u_s^{k*}(y), \quad (6.71)$$

and using this formula and (6.13), we obtain the following expression for the modular Hamiltonian kernel

$$\mathcal{K}(x, y) = \sum_{k=1}^n \int_{-\infty}^{+\infty} ds u_s^k(x) 2\pi s u_s^{k*}(y). \quad (6.72)$$

Using equation (6.60), we get

$$\mathcal{K}(x, y) = -i2\pi k(x, y) \delta'(\omega(x) - \omega(y)), \quad (6.73)$$

where

$$k(x, y) = 2\pi \sqrt{\prod_{i=1}^n \frac{(x-a_i)(y-a_i)}{(x-b_i)(y-b_i)}} \left(\sum_{k=1}^n \frac{1}{N_k^2 (x-a_k)(y-a_k)} \right). \quad (6.74)$$

The aim of the rest of this subsection is to obtain a simplified expression for the modular Hamiltonian kernel (6.73). First, we have the following identity for the Dirac delta term

$$\delta'(\omega(x) - \omega(y)) = \sum_{l=1}^n \delta'(x-x_l) \frac{1}{\omega'(x)^2} - \delta(x-x_l) \frac{\omega''(x)}{\omega'(x)^3}, \quad (6.75)$$

where $x_l \equiv x_l(y) \in A_l$ are the roots of $\omega(x) = \omega(y)$ introduced in equation (6.64). From this last equation, the modular Hamiltonian splits into the sum of a local and a non-local kernel

$$\mathcal{K}(x, y) = \mathcal{K}_{loc}(x, y) + \mathcal{K}_{noloc}(x, y), \quad (6.76)$$

where $\mathcal{K}_{loc}(x, y)$ comes from the term in (6.75) with $x_l(y) = y$, and $\mathcal{K}_{noloc}(x, y)$ comes from the $n-1$ terms in (6.75) with $x_l(y) \neq y$. We discuss these two contributions separately.

Local part

The local part for the modular Hamiltonian kernel is

$$\mathcal{K}_{loc}(x, y) = -i2\pi k(x, y) \left[\frac{1}{\omega'(x)^2} \delta'(x-y) - \frac{\omega''(x)}{\omega'(x)^3} \delta(x-y) \right]. \quad (6.77)$$

In order to simplify the above expression, we need to understand (6.77) as a distribution acting over some smooth compactly supported test function $\varphi(x, y)$. Integrating by parts,

the derivative of the Dirac delta is converted to

$$\begin{aligned} \varphi(x, y)k(x, y)\frac{1}{\omega'(x)^2}\delta'(x - y) = & - \left[\partial_x \varphi(x, y)k(x, x)\frac{1}{\omega'(x)^2} + \varphi(x, x) \partial_x k(x, y)|_{y=x} \frac{1}{\omega'(x)^2} \right. \\ & \left. + \varphi(x, y)k(x, x)\partial_x \left(\frac{1}{\omega'(x)^2} \right) \right] \delta(x - y), \end{aligned} \quad (6.78)$$

The above expression can be simplified by using the following identities

$$k(x, x) = \omega'(x), \quad (6.79)$$

$$\partial_x k(x, y)|_{y=x} = \frac{1}{2}\omega''(x). \quad (6.80)$$

which follow from (6.74) and the algebraic relation¹⁴

$$\sum_{k=1}^n \frac{1}{N_k^2} \frac{1}{(x - a_k)} = \frac{1}{2\pi} (1 + e^{-\omega(x)}). \quad (6.81)$$

The final steps consist on replacing (6.79) and (6.80) into (6.78), and integrating by parts the term containing $\partial_x \varphi(x, y)$, in order to factorize the test function. We finally get

$$\mathcal{K}_{loc}(x, y) = -i2\pi \left[\frac{1}{\omega'(x)}\delta'(x - y) + \frac{1}{2}\partial_x \left(\frac{1}{\omega'(x)} \right) \delta(x - y) \right]. \quad (6.82)$$

The local part of the modular Hamiltonian comes from (6.12) and (6.82)

$$K_{loc} = 2\pi \int_A dx \frac{1}{\omega'(x)} T(x), \quad (6.83)$$

where $T(x) = \frac{1}{2} [i\partial_x \psi^\dagger(x)\psi(x) - \psi^\dagger(x)i\partial_x \psi(x)]$ is the energy density operator.

Non-local part

The non-local part of the modular Hamiltonian kernel is

$$\mathcal{K}_{nonloc}(x, y) = -i2\pi k(x, y) \left[\sum_{l=1}^n \frac{1}{\omega'(x)^2} \delta'(x - x_l) - \frac{\omega''(x)}{\omega'(x)^3} \delta(x - x_l) \right]. \quad (6.84)$$

The first term can be simplified by a similar computation as we did around equation (6.78). Here, the situation is simpler because $k(x_l, y) \equiv 0$ for all $x_l \neq y$ as we showed in (6.65). Hence, the unique term which survives is the one proportional to the derivative of $k(x, y)$,

¹⁴Like (6.65), equation (6.81) is a pure algebraic relation valid for any complex number a_k , b_k and x . It can be shown matching the coefficients of the poles in x on both sides.

i.e,

$$\partial_x k(x, y)|_{x=x_l} = \frac{\omega'(x_l)}{x_l - y}. \quad (6.85)$$

Replacing it on (6.84), we arrive to

$$\mathcal{K}_{noloc}(x, y) = i2\pi \sum_{l=1, x_l \neq y}^n \frac{1}{(x-y)} \frac{1}{\omega'(x)} \delta(x - x_l(\omega(y))), \quad (6.86)$$

and

$$K_{noloc} = i2\pi \sum_{l=1, x_l \neq x}^n \int_A dx \psi^\dagger(x_l) \frac{1}{(x_l - x)} \frac{1}{\omega'(x_l)} \psi(x) \quad (6.87)$$

$$= i2\pi \int_{A \times A, x \neq y} dx dy \psi^\dagger(x) \frac{\delta(\omega(x) - \omega(y))}{x - y} \psi(y). \quad (6.88)$$

Two intervals case

For the case of two intervals $A = (a_1, b_1) \cup (a_2, b_2)$, the modular Hamiltonian operator $K = K_{loc} + K_{noloc}$ reduces to

$$K_{loc} = 2\pi \int_A dx \omega'(x)^{-1} T(x), \quad (6.89)$$

$$K_{noloc} = i2\pi \int_A dx \gamma(x) \psi^\dagger(x) \psi(\bar{x}), \quad (6.90)$$

where

$$\omega'(x) = \frac{1}{x - a_1} + \frac{1}{x - a_2} - \frac{1}{x - b_1} - \frac{1}{x - b_2}, \quad (6.91)$$

$$\gamma(x) = \frac{\omega'(x)^{-2}}{x(a_1 + a_2 - b_1 - b_2) + (b_1 b_2 - a_1 a_2)} \frac{(b_1 - a_1)(a_2 - b_1)(b_2 - a_1)(b_2 - a_2)}{(x - a_1)(x - a_2)(x - b_1)(x - b_2)}, \quad (6.92)$$

$$\bar{x} = \frac{a_1 a_2 (x - b_1 - b_2) - b_1 b_2 (x - a_1 - a_2)}{x(a_1 + a_2 - b_1 - b_2) + (b_1 b_2 - a_1 a_2)}. \quad (6.93)$$

For two symmetric intervals $A = (-R, -r) \cup (r, R)$, $0 < r < R$, we can write a more explicit form of the modular Hamiltonian

$$K = 2\pi \int_A dx \frac{(x^2 - r^2)(R^2 - x^2)}{2(R - r)(x^2 + rR)} T(x) + i\pi \int_A dx \psi^\dagger(x) \frac{rR(x^2 - r^2)(R^2 - x^2)}{(R - r)x(x^2 + rR)^2} \psi(\bar{x}), \quad (6.94)$$

where now $\bar{x} = -\frac{rR}{x}$.

6.3. The free chiral current

The free chiral current field is defined by means a field operator $J(x)$ in the null-line, which is identified with the chiral derivative of a massless free scalar in $d = 2$, i.e. $J(x^+) = \partial_+ \phi(x^+)$. It is usually defined as a quantum field $J(x)$ satisfying the commutation relations

$$[J(x), J(y)] = i \delta'(x - y) =: i C(x - y), \quad (6.95)$$

and having the vacuum two-point correlator

$$F(x, y) := \langle J(x) J(y) \rangle = -\frac{1}{2\pi} \frac{1}{(x - y - i0^+)^2}. \quad (6.96)$$

This model is Gaussian and all other multipoint correlators follow from (6.96) according to Wick's theorem. From the algebraic standpoint, given a test function $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$, the operator

$$J(f) := \int_{\mathbb{R}} J(x) f(x) dx, \quad (6.97)$$

is a well-defined unbounded operator acting on the bosonic Fock Hilbert space \mathcal{H}_0 . This Hilbert space is constructed as the symmetric tensor product of the one-particle Hilbert space of massless particles of zero helicity. This one-particle Hilbert space carries a unitary representation of $\mathbf{M\ddot{o}b}$, which naturally extends to a unitary representation of $\mathbf{M\ddot{o}b}$ on \mathcal{H}_0 . The chiral Hamiltonian can be expressed as

$$H = \frac{1}{2} \int dx : J^2(x) :, \quad (6.98)$$

where $T(x) := \frac{1}{2} : J^2(x) :$ is the energy density operator. In order to get bounded operators, we define the Weyl unitaries $W(f) := e^{iJ(f)}$, which satisfy the usual CCR relations

$$W(f) W(g) = e^{i \int_{\mathbb{R}} dx f'(x) g(x)} W(g) W(f). \quad (6.99)$$

For any open set $A \subset \mathbb{R}$, the local algebras are defined as

$$\mathcal{A}(A) := \{e^{iJ(f)} : \text{supp}(f) \subset A\}'' . \quad (6.100)$$

It is not difficult to show that such a collection of algebras satisfies all the axioms of the definition 6.1.

It is interesting to note that the free chiral current is a subnet of the free chiral fermion. In fact, the current fermion operator $J_\psi(x) :=: \psi^\dagger(x) \psi(x) :$ satisfies the commutation relations (6.95), and has the same vacuum expectation value (6.96). More precisely, the

net of vN algebras

$$\mathcal{B}_f(A) := \left\{ e^{i \int_{\mathbb{R}} J_\psi(x) f(x)} : \text{supp}(f) \subset A \right\}'' \subsetneq \mathcal{A}_f(A) \quad (6.101)$$

is a subnet of the free chiral fermion isomorphic to the current net 6.100 [131, 132].

In the following, we study the modular Hamiltonian associated with the vacuum state and a region $A \subset \mathbb{R}$. Because of the complexity of the problem, we restrict the attention to the case of one or two intervals. The commutation relations (6.95) define a general CCR algebra, and hence, we can use all the formulas developed in section 4.3.3. Then, the modular Hamiltonian is given by equations (4.86), (4.89), and (4.84), which means

$$K = \int_{A \times A} dx dy J(x) \mathcal{K}(x, y) J(y), \quad (6.102)$$

$$\mathcal{K} = -\frac{i}{2} \frac{V}{|V|} \log \left(\frac{|V| + 1/2}{|V| - 1/2} \right) C^{-1}, \quad (6.103)$$

$$V = -iC^{-1}F - \frac{1}{2}. \quad (6.104)$$

We emphasize that the operators \mathcal{K}, V, C , and F above are operators acting as kernels in the Hilbert space $L^2(A)$ of square-integrable functions with support in A . According to the calculations from the lattice model, the operator V is not self-adjoint, but it is diagonalizable, and its spectrum belongs to $(-\infty, -1/2] \cup [1/2, +\infty)$. In the continuum QFT, we expect that the operator V has a complete set of (generalized) eigenvectors, with continuous spectrum $\text{spec}(V) = (-\infty, -1/2] \cup [1/2, +\infty)$. For later convenience, we parametrize the corresponding eigenvalues by $\frac{1}{2} \coth(\pi s)$ with $s \in \mathbb{R}$. The eigenvectors of V and V^\dagger are denoted by

$$V|u_s^k\rangle = \frac{1}{2} \coth(\pi s)|u_s^k\rangle, \quad (6.105)$$

$$V^\dagger|v_s^k\rangle = \frac{1}{2} \coth(\pi s)|v_s^k\rangle, \quad (6.106)$$

where $k \in \mathcal{Y}$ is a possible degeneracy index. We normalize the eigenvectors according to

$$\langle v_s^k | u_{s'}^{k'} \rangle = \int_A v_s^k(x)^* u_{s'}^{k'}(x) dx = \delta_{k,k'} \delta(s - s'). \quad (6.107)$$

It is not difficult to see, from (6.104), (6.105), and (6.106), that $C|u_s^k\rangle$ is an eigenvector of V^\dagger with eigenvalue $\frac{1}{2} \coth(\pi s)$, and hence, it is a linear combination of the vectors $\{|v_s^k\rangle : k \in \mathcal{Y}\}$. The orthogonality relation (6.107) leaves us the freedom to redefine the eigenbasis $\{|u_s^k\rangle : k \in \mathcal{Y}\}$ by an arbitrary matrix, and the eigenbasis $\{|v_s^k\rangle : k \in \mathcal{Y}\}$ by the inverse adjoint matrix. We can use this freedom to set $|v_s^k\rangle \propto C|u_s^k\rangle$ for all $s \in \mathbb{R}$ and

all $k \in \mathcal{Y}$. The phase of the proportionality constant is fixed to be $i \operatorname{sign}(s)$, as we can see from taking the scalar product of (6.105) with $\langle v_s^{k'} |$ and using (6.107) and the positivity of F . As a result, we can further fix the eigenvectors by taking

$$|v_s^k\rangle := i \operatorname{sign}(s) C |u_s^k\rangle. \quad (6.108)$$

In terms of these vectors, the kernel (6.103) is written simply as

$$\mathcal{K}(x, y) = \sum_{k \in \mathcal{Y}} \int_{-\infty}^{+\infty} ds u_s^k(x)^* \pi |s| u_s^k(y). \quad (6.109)$$

The EE and the Rényi EE are given by formulas (4.87) and (4.88). They could be rewritten in terms of the eigenvectors (6.105-6.106) as

$$S(A) = \sum_{k \in \mathcal{Y}} \int_0^{+\infty} ds \int_A dx g(s) u_s^k(x) v_s^k(x)^*, \quad (6.110)$$

$$S_\alpha(A) = \sum_{k \in \mathcal{Y}} \int_0^{+\infty} ds \int_A dx g_\alpha(s) u_s^k(x) v_s^k(x)^*, \quad (6.111)$$

where

$$g(s) := \frac{1 + \coth(\pi s)}{2} \log \left(\frac{\coth(\pi s) + 1}{2} \right) + \frac{1 - \coth(\pi s)}{2} \log \left(\frac{\coth(\pi s) - 1}{2} \right), \quad (6.112)$$

$$g_\alpha(s) := \frac{1}{\alpha - 1} \log \left[\left(\frac{\coth(\pi s)}{2} + \frac{1}{2} \right)^\alpha - \left(\frac{\coth(\pi s)}{2} - \frac{1}{2} \right)^\alpha \right]. \quad (6.113)$$

The integrals over the x coordinate in equations (6.110) and (6.111) give delta functions $\delta(0)$, and hence, the EE and the Rényi EE are divergent. This is just the usual divergence in QFT due to the continuum spectrum of the modular Hamiltonian. A convenient way to regularize these entropies is to integrate up to a distance cutoff $\epsilon > 0$ from the boundary ∂A (see section 4.3.4). In the two interval case, the integration region A has to be replaced by $A^{(\epsilon)} := (a_1 + \epsilon, b_1 - \epsilon) \cup (a_2 + \epsilon, b_2 - \epsilon)$.

In sections 6.3.1 and 6.3.2, we handle the case of one and two intervals independently.

6.3.1. The single interval case

Let us first consider the simplest case of a single interval $A := (a, b)$. The inverse $(\delta')^{-1}(x, y)$ of the commutator $\delta'(x - y)$, acting on a test function $f(x)$, has to be a linear combination of

$$\int_a^b dy (\delta')^{-1}(x, y) f(y) = \int_a^x dy f(y) + c \int_a^b dy l(y) f(y), \quad (6.114)$$

where $c \in \mathbb{C}$ and $l(y)$ is some fixed function. This last term is linear in $f(y)$, and since it is independent of x , it is annihilated by δ' . In kernel notation we have to combine¹⁵

$$\Theta(x-y), \quad -\Theta(y-x), \quad l(y). \quad (6.115)$$

There is only one antisymmetric inverse, given by

$$(\delta')^{-1}(x, y) := \frac{1}{2} (\Theta(x-y) - \Theta(y-x)), \quad (6.116)$$

which acts on a test function $f(x)$ as

$$((\delta')^{-1}f)(x) = \frac{1}{2} \left(\int_a^x dy f(y) - \int_x^b dy f(y) \right). \quad (6.117)$$

Then, we have that $\delta' \circ (\delta')^{-1} = \delta$, and $(\delta')^{-1} \circ \delta' = \delta$ on test functions that vanish on the boundary of the interval.

Hence, following (6.104-6.106) and (6.116), our first task is to solve the spectrum of

$$2\pi C^{-1}F = \frac{1}{(x-y-i0^+)} - \frac{1}{2(b-y)} - \frac{1}{2(a-y)}, \quad (6.118)$$

and

$$2\pi FC^{-1} = \frac{1}{(x-y-i0^+)} - \frac{1}{2(b-x)} - \frac{1}{2(a-x)}. \quad (6.119)$$

6.3.1.1. A complete set of eigenvectors

The aim of this subsection is to find a complete set of the eigenvectors for the kernels (6.118) and (6.119), which are the eigenvectors of the relevant kernels (6.105) and (6.106) in the single interval case. In the end, the eigenvectors are given by (6.138) and (6.139) below.

We proceed as in the case of the fermion field. We think the interval A as included in the real axis of the complex plane, and we consider an analytic function $S(z)$, having a multiplicative boundary condition on the interval A as in (6.15), i.e.

$$S(z) \text{ analytic in } \mathbb{C} - \bar{A}, \quad (6.120)$$

$$S^+(x_1) = \lim_{x_2 \rightarrow 0^+} S(x_1 + ix_2) = \lambda \lim_{x_2 \rightarrow 0^-} S(x_1 + ix_2) = \lambda S^-(x_1), \quad x_1 \in A. \quad (6.121)$$

¹⁵Only two of these three kernels are linearly independent.

We further impose boundary conditions at infinity and at the endpoints of the interval,

$$\lim_{z \rightarrow \infty} |S(z)| < \infty, \quad (6.122)$$

$$\lim_{z \rightarrow \partial A} |S(z)| < \infty. \quad (6.123)$$

Consider now the complex integral

$$\oint dz_2 \left(\frac{1}{z_1 - z_2} - \frac{1}{2} \frac{1}{b - z_2} - \frac{1}{2} \frac{1}{a - z_2} \right) S(z_2) = 0 \quad (6.124)$$

along a closed curve in the complex plane encircling both A and $z_1 \in \mathbb{C} - \bar{A}$. This integral is zero because the integrand has zero residue at infinity. We can shrink the integration contour around the point z_1 and around the cut to get

$$S(z_1) = \frac{i}{2\pi} (1 - \lambda^{-1}) \int_A dy \left(\frac{1}{z_1 - y} - \frac{1}{2} \frac{1}{b - y} - \frac{1}{2} \frac{1}{a - y} \right) S^+(y). \quad (6.125)$$

The symbol \int for the integral means that it is regularized at the endpoints of the interval by taking the complex integral on a small circle around these endpoints (as implied by (6.124)), and then take the limit of zero circle size. We will soon be more specific on how this regularization can be expressed directly by means the function $S^+(y)$.

Taking the limit of $z_1 \rightarrow x - i0^+$ with $x \in A$, and using (6.121), (6.118), and (6.104), we arrive to

$$\int_A dy V(x, y) S^+(y) = \frac{\lambda + 1}{2(1 - \lambda)} S^+(x). \quad (6.126)$$

Since the eigenvalues of $|V|$ are in $[1/2, \infty)$, we have that $\lambda > 0$ in contrast to the case of the fermion, where λ is negative. We then write

$$\lambda := e^{-2\pi s}, \quad s \in \mathbb{R}. \quad (6.127)$$

The eigenvalue in (6.126) then coincides with $\frac{1}{2} \coth(\pi s)$ as in (6.105).

Therefore, for each solution $S(z)$ of the problem (6.120-6.123) in the complex plane, we get an eigenvector of the kernel $V(x, y)$ on the interval A . The eigenvectors, in contrast to the case of the fermion, are bounded at the endpoints of the interval (see equation (6.123)). This is in accordance with boundary conditions for scalars [105]. Conversely, if we have a solution of (6.126), we can use it as boundary data on the interval in (6.125), which gives a solution $S(z)$ satisfying (6.120-6.123). These problems are then mutually equivalent.

For the single interval A , we can write a solution to the above problem as

$$S(z) := e^{-is\tilde{\omega}(z)}, \quad \tilde{\omega}(z) := \log \left(\frac{z - a}{z - b} \right). \quad (6.128)$$

This obeys all the conditions (6.120–6.123). To show that (6.128) is the unique solution we proceed as follows. Let us suppose that $\tilde{S}(z)$ is another solution to the same problem. Then, the function $\tilde{S}(z)S(z)^{-1}$ is an analytic function on the plane except, perhaps, at the endpoints of the interval. But, because of the boundary conditions (6.122–6.123), we must have that $\tilde{S}(z)S(z)^{-1}$ is bounded at infinity and at the endpoints of the interval. Then, we have that $\tilde{S}(z)S(z)^{-1}$ is a constant function. Therefore, (6.128) is the unique solution to the problem.

The eigenvectors are given by the boundary values of $S(z)$ on the interval

$$u(x) \propto S^+(x) \propto e^{-is\omega(x)}, \quad \omega(x) := \log\left(\frac{x-a}{b-x}\right). \quad (6.129)$$

Now we explain more precisely the regularization in (6.125) and (6.126). Frequently, we will encounter integrals on the real line of the form

$$\int_a^b dx f(x), \quad \text{where } f(x) \sim c \frac{e^{-is \log(x-a)}}{x-a}, \quad x \rightarrow a^+. \quad (6.130)$$

Then, this integral will have an oscillatory but bounded term $c e^{-is \log(x-a)}/(is)$ in the lower boundary as $x \rightarrow a^+$, and hence, it does not converge. The regularization used above just subtracts this oscillatory phase, i.e.

$$\begin{aligned} \int_a^b dx f(x) &= \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^b dx f(x) - c \frac{e^{-is\epsilon}}{is} \\ &= \int_a^b dx \left(f(x) - c \frac{e^{-is \log(x-a)}}{x-a} \right) + c \frac{e^{-is \log(b-a)}}{(-is)}. \end{aligned} \quad (6.131)$$

If the oscillatory term appears on the upper end of the integral, an analogous subtraction is understood. As we mentioned above, this subtraction appears naturally when the integral comes from a limit of a complex integral around the cut, as in the transformation from (6.124) to (6.125). The definition of the kernel V has to be understood with this regularization.¹⁶

Now, we have to look at the eigenvectors of (6.106). For this, we can just use the relation (6.108). However, we find instructive to compute them directly from the kernel (6.119). To do that, we take a different analytic function $S(z)$ and assume that it satisfies the same multiplicative boundary condition (6.121). However, in order to obtain a solution

¹⁶Note that this regularization eliminates from the bare integral an infinitely oscillatory phase in s , which produces vanishing terms in any finite calculation involving integrals over the spectrum.

of the eigenvector problem from the complex integral, we are now forced to impose

$$S(z) \sim \frac{1}{|z|^2}, \quad z \rightarrow \infty, \quad (6.132)$$

$$S(z) \text{ have at most pole singularities at } a \text{ and } b. \quad (6.133)$$

As above, we consider the complex integral

$$\oint dz_2 \left(\frac{1}{z_1 - z_2} - \frac{1}{2} \frac{1}{b - z_1} - \frac{1}{2} \frac{1}{a - z_1} \right) S(z_2), \quad (6.134)$$

along a closed curve in the complex plane encircling both A and $z_1 \in \mathbb{C} - \bar{A}$. We shrink the integration contour around the point z_1 and around the cut to get

$$S(z_1) = \frac{i}{2\pi} (1 - \lambda^{-1}) \int_A dy \left(\frac{1}{z_1 - y} - \frac{1}{2} \frac{1}{b - z_1} - \frac{1}{2} \frac{1}{a - z_1} \right) S^+(y). \quad (6.135)$$

The limit $z_1 \rightarrow x - i0^+$, with $x \in A$, gives

$$\int_A dy V^\dagger(x, y) S^+(y) = \frac{\lambda + 1}{2(1 - \lambda)} S^+(x). \quad (6.136)$$

The value of λ is the same as in (6.127), giving the same multiplicative boundary conditions for $S(z)$ as for the eigenvectors of V . However, the boundary conditions at infinity and at the endpoints of the interval are different. These imply that the unique solution is

$$S(z) := e^{-is\bar{w}(z)} \left(\frac{1}{z - a} - \frac{1}{z - b} \right). \quad (6.137)$$

The poles have to have opposite signs so that function decays at infinity as $|z|^{-2}$. We recognize this function is proportional to the derivative of (6.128), and it must be because of (6.108).

To end, we obtain the eigenvectors

$$u_s(x) = \frac{e^{-is\omega(x)}}{\sqrt{2\pi|s|}}, \quad (6.138)$$

$$v_s(x) = i \operatorname{sign}(s) u'_s(x) = \sqrt{\frac{|s|}{2\pi}} e^{-is\omega(x)} \left(\frac{1}{x - a} - \frac{1}{x - b} \right), \quad (6.139)$$

which are normalized in order to satisfy (6.107) and (6.108).

6.3.1.2. Modular Hamiltonian

Replacing the formula (6.138) for the eigenvectors into the equation (6.109) and after a simple integration, we get the following expression for the modular Hamiltonian kernel

$$\mathcal{K}(x, y) = \int_{-\infty}^{\infty} ds u_s(x) \pi |s| (u_s(y))^* = \pi (\omega'(x))^{-1} \delta(x - y). \quad (6.140)$$

Then, the modular Hamiltonian operator has the well-known form for an interval in a CFT [34, 35]

$$K_A = 2\pi \int_a^b dx \frac{(b-x)(x-a)}{b-a} T(x), \quad (6.141)$$

where in the present case the energy density operator is $T(x) := \frac{1}{2} : J^2(x) :$

6.3.1.3. Entanglement entropy

According to (6.110) and (6.112), the EE is

$$S(A) = \int_{a+\epsilon}^{b-\epsilon} dx \int_0^{\infty} ds g(s) u_s(x) (v_s(x))^* = \frac{1}{12} \int_{a+\epsilon}^{b-\epsilon} dx \omega'(x) = \frac{1}{6} \log \left(\frac{b-a}{\epsilon} \right). \quad (6.142)$$

This gives the expected result for a conformal model with one chiral component of central charge $c = 1$. The Rényi EE can be computed analogously using (6.111) and (6.113)

$$S_\alpha(A) = \frac{1+\alpha}{12\alpha} \log \left(\frac{b-a}{\epsilon} \right). \quad (6.143)$$

6.3.2. The two interval case

Now, we consider a two-interval region $A := A_1 \cup A_2$, where $A_i := (a_i, b_i)$. To start, we need first to know the expression of the kernel $C^{-1} = (\delta')^{-1}$ for two intervals. The commutator is block diagonal in each of the intervals, and we get the same result as (6.116) for each of the intervals separately

$$C^{-1}(x, y) := \begin{cases} \frac{1}{2} (\Theta(x-y) - \Theta(y-x)), & \text{if } x, y \in A_1 \text{ or } x, y \in A_2, \\ 0, & \text{otherwise,} \end{cases} \quad (6.144)$$

or equivalently,

$$C^{-1}(x, y) = \frac{1}{2} (\Theta(x-y) - \Theta(y-x)) - \frac{1}{2} \Theta(x-a_2) \Theta(b_1-y) + \frac{1}{2} \Theta(y-a_2) \Theta(b_1-x). \quad (6.145)$$

Notice that this last expression is antisymmetric, and its derivative is the delta function. Then, we have that

$$2\pi C^{-1}F = \frac{1}{(x-y-i0^+)} + \frac{1}{2} \left(\Theta_{A_1}(x) \left(\frac{1}{y-a_1} + \frac{1}{y-b_1} \right) + \Theta_{A_2}(x) \left(\frac{1}{y-a_2} + \frac{1}{y-b_2} \right) \right), \quad (6.146)$$

and

$$2\pi FC^{-1} = \frac{1}{(x-y-i0^+)} - \frac{1}{2} \left(\left(\frac{1}{x-a_1} + \frac{1}{x-b_1} \right) \Theta_{A_1}(y) + \left(\frac{1}{x-a_2} + \frac{1}{x-b_2} \right) \Theta_{A_2}(y) \right), \quad (6.147)$$

where $\Theta_{A_j}(x) = \Theta(x-a_j)\Theta(b_j-x)$ is the characteristic functions of the region A_j .

6.3.2.1. A complete set of eigenvectors

The aim of this subsection is to find a complete set of the eigenvectors for the kernels (6.146) and (6.147), which are the eigenvectors of the relevant kernels (6.105) and (6.106) in the two intervals case. In the end, the eigenvectors are given by (6.167) and (6.170) below.

Now, we have to deal with the kernels (6.146) and (6.147) which contain theta functions. At first glance, it might seem that the analytic method used in previous sections does not apply here. However, we will show how to bypass this issue. To begin with, let us consider eigenvectors $v_s(x)$ of (6.147) satisfying the following extra property

$$\oint_{A_1} dx v_s(x) = \oint_{A_2} dx v_s(x) = 0. \quad (6.148)$$

For such particular eigenvectors, the second and third terms in (6.147) vanish. In the end, we will show that (6.148) is true in the general case. Under this assumption, we have that $v_s(x)$ is an eigenfunction of $(x-y-i0^+)^{-1}$. Then we use the same ideas as for a single interval, trying to obtain $v_s(x)$ as a boundary value of an analytic function. We again look for analytic functions $S(z)$ on the complex plane with multiplicative boundary conditions on the two intervals A as in (6.121). The class of eigenfunctions $u_s(x)$ of the problem must behave near the endpoints of the intervals as in the single interval case. That is, they should behave as pure phase factors of the form

$$u_s(x) \sim \begin{cases} e^{-is \log(x-a_j)}, & x \rightarrow a_j^+, \\ e^{is \log(b_j-x)}, & x \rightarrow b_j^-. \end{cases} \quad (6.149)$$

Their derivatives, the functions $v_s(x)$, should have at most single poles (together with a phase factor) at the interval endpoints. Under these conditions, the most general solution is of the form

$$S(z) \propto e^{-is\tilde{\omega}(z)} \left(\frac{\alpha_1}{z-a_1} + \frac{\alpha_2}{z-b_1} + \frac{\alpha_3}{z-a_2} + \frac{\alpha_4}{z-b_2} \right), \quad (6.150)$$

with $\alpha_l = \alpha_l(s; a_1, b_1, a_2, b_2)$ ($l = 1, 2, 3, 4$) and

$$\tilde{\omega}(z) := \sum_{j=1}^2 \log \left(\frac{z-a_j}{z-b_j} \right). \quad (6.151)$$

Integrating $S(z)$ along contours encircling the two intervals and a large circle at infinity, it is not difficult to see that the integral at infinity is equal to the one over the two intervals, which vanishes because of (6.148). Then, this function must fall as $|z|^{-2}$ to cope with (6.148), and we must impose the condition

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0. \quad (6.152)$$

Calling $\vec{q} := (a_1, b_1, a_2, b_2)$, we have from (6.148) that the coefficients α_j satisfy in addition

$$\sum_{j=1}^4 \alpha_j I_{q_j}^1 = 0, \quad (6.153)$$

$$\sum_{j=1}^4 \alpha_j I_{q_j}^2 = 0, \quad (6.154)$$

where

$$I_{q_j}^l = \int_{a_l}^{b_l} dx e^{-is\omega(x)} \frac{1}{x - q_j}, \quad l = 1, 2, \quad j = 1, 2, 3, 4, \quad (6.155)$$

and

$$\omega(x) := \log \left(-\frac{(x-a_1)(x-a_2)}{(x-b_1)(x-b_2)} \right). \quad (6.156)$$

Only two of the equations (6.152), (6.153), and (6.154) are independent. This follows from the fact that

$$\oint dz e^{-is\omega(z)} \left(\frac{1}{z-q_j} - \frac{1}{z-q_k} \right) = I_{q_j}^1 - I_{q_k}^1 + I_{q_j}^2 - I_{q_k}^2 = 0. \quad (6.157)$$

This complex integral around the two cuts is zero because it is equal to the integral at infinity, which vanishes because the integrand falls fast enough. Therefore, we can conclude that the dimension of the space of solutions for fixed s is 2. The same argument of the previous section shows that these solutions give the eigenvectors of V^\dagger once evaluated on A . Conversely, any eigenvector of V^\dagger , with at most simple poles at the end of the intervals

and satisfying (6.148), is of this form.

Now a simple solution is

$$\tilde{v}_1(z) \propto \frac{d}{dz} e^{-is\tilde{\omega}(z)} = -i s e^{-is\tilde{\omega}(z)} \left(\frac{1}{z-a_1} - \frac{1}{z-b_1} + \frac{1}{z-a_2} - \frac{1}{z-b_2} \right). \quad (6.158)$$

In fact, this satisfies (6.152) and (6.153) because it is proportional to a derivative of the phase $e^{-is\tilde{\omega}(z)}$ and hence, the integral on any of the intervals vanish with the regularization we are using. That is, integrating this function along the intervals we have further relations for the integrals I_q^l

$$I_{a_1}^1 - I_{b_1}^1 + I_{a_2}^1 - I_{b_2}^1 = 0, \quad (6.159)$$

$$I_{a_1}^2 - I_{b_1}^2 + I_{a_2}^2 - I_{b_2}^2 = 0. \quad (6.160)$$

The eigenvector $v_1(x)$ follows from taking the boundary limit of $\tilde{v}_1(z)$ on A from above. The corresponding $u_1(x)$ eigenfunction is an integral of this function,

$$u_1(x) = -i \operatorname{sign}(s) (C^{-1} v_1)(x) \propto e^{-is\omega(x)}, \quad (6.161)$$

where in applying (6.145) to $v_1(x)$, boundary terms that are oscillatory phases are discarded, in accordance with the regularization (6.131). We can check directly that (6.161) is an eigenfunction of (6.146) by noting that

$$\int_A dy \left(\frac{1}{y-a_1} + \frac{1}{y-b_1} \right) u_1(y) = \int_A dy \left(\frac{1}{y-a_2} + \frac{1}{y-b_2} \right) u_1(y). \quad (6.162)$$

This follows from (6.157), (6.159), and (6.160). Hence, the two terms involving the characteristic functions in (6.146) are equal, and we can eliminate these functions altogether by replacing $\Theta_{A_1}(x), \Theta_{A_2}(x) \rightarrow \frac{1}{2}$. After these replacements, the proof that $u_1(x)$ is an eigenvector of the kernel (6.146) follows from the same steps as the one for a single interval in the previous section, by promoting $u_1(x)$ to a complex function $\tilde{u}_1(z) \propto e^{-is\tilde{\omega}(z)}$.

We choose the second solution $\tilde{v}_2(z)$ of the form (6.150), such that its boundary value on A gives a $v_2(x)$ eigenfunction orthogonal to $u_1(x)$. Collecting the coefficients of the would-be delta functions in the scalar product between $u_1(x)$ and $v_2(x)$, which are generated by the integral near the intervals endpoints, we have for $\tilde{v}_2(z)$

$$\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = 0. \quad (6.163)$$

From this and (6.152), we get

$$\tilde{v}_2(z) \propto e^{-isw(z)} \left(\frac{1}{z-a_1} + \frac{\alpha}{z-b_1} - \frac{1}{z-a_2} - \frac{\alpha}{z-b_2} \right). \quad (6.164)$$

And from (6.153), it follows that

$$\alpha(s; a_1, b_1, a_2, b_2) = -\frac{I_{a_1}^1 - I_{a_2}^1}{I_{b_1}^1 - I_{b_2}^1}. \quad (6.165)$$

In order to compute $u_2(x)$ we use (6.148). Therefore, using (6.108) and (6.147), we have that

$$u_2(x) = -i \operatorname{sign}(s) \begin{cases} \int_{a_1}^x dy v_2(y) = \int_{b_1}^x dy v_2(y), & x \in A_1 \\ \int_{a_2}^x dy v_2(y) = \int_{b_2}^x dy v_2(y), & x \in A_2 \end{cases}. \quad (6.166)$$

In order to normalize the solutions, we compute the coefficient of the delta function in the scalar product, which can only come from the singular part of the integrals near the endpoints of the intervals. We have that the normalized solutions satisfying (6.108) are

$$v_1(x) = \sqrt{\frac{|s|}{4\pi}} e^{-is\omega(x)} \left(\frac{1}{x-a_1} - \frac{1}{x-b_1} + \frac{1}{x-a_2} - \frac{1}{x-b_2} \right), \quad (6.167)$$

$$u_1(x) = \frac{1}{\sqrt{4\pi|s|}} e^{-is\omega(x)}, \quad (6.168)$$

$$v_2(x) = \sqrt{\frac{|s|}{4\pi}} e^{-is\omega(x)} \left(\frac{1}{x-a_1} + \frac{\alpha}{x-b_1} - \frac{1}{x-a_2} - \frac{\alpha}{x-b_2} \right), \quad (6.169)$$

$$u_2(x) = \frac{-is}{\sqrt{4\pi|s|}} \begin{cases} \int_{a_1}^x dy e^{-is\omega(y)} \left(\frac{1}{y-a_1} + \frac{\alpha}{y-b_1} - \frac{1}{y-a_2} - \frac{\alpha}{y-b_2} \right), & x \in A_1 \\ \int_{a_2}^x dy e^{-is\omega(y)} \left(\frac{1}{y-a_1} + \frac{\alpha}{y-b_1} - \frac{1}{y-a_2} - \frac{\alpha}{y-b_2} \right), & x \in A_2 \end{cases}. \quad (6.170)$$

Completeness of the eigenvector system

Now we show the eigenvector basis (6.167-6.170) is complete. When we explicitly constructed the eigenvectors, we only consider solutions satisfying the equation (6.148) in order to simplify the calculation, but there was no further reason to assume that. Now, we are able to show that any other possible eigenvector must satisfy (6.148), and hence, there are no other eigenvectors than the ones already obtained. This fact follows by considering the $s = 0$ solutions for $u_1(x)$ and $u_2(x)$. Taking out an irrelevant factor of $|s|^{-1/2}$, which is compensated by the inverse factor in the eigenfunctions $v(x)$, we have that

$$\lim_{s \rightarrow 0} \sqrt{4\pi|s|} u_{1,s}(x) = 1, \quad (6.171)$$

$$\lim_{s \rightarrow 0} \sqrt{4\pi|s|} u_{2,s}(x) = \Theta_{A_1}(x) - \Theta_{A_2}(x). \quad (6.172)$$

The first one is proportional to a constant, which is the same along the two intervals. The second one is proportional to two opposite constants in the two different intervals. Hence, any third solution $v_{3,s}(x)$ would be orthogonal to $u_{1,s=0}(x)$ and $u_{2,s=0}(x)$, and therefore must satisfy (6.148). Therefore, there cannot be any other eigenvectors for two intervals.

6.3.2.2. Dependence of the eigenvectors through the cross ratio

The aim of this subsection is to obtain simplified expressions for the function (6.165) and the eigenfunctions (6.167–6.170), which will be useful for the final computation of the modular Hamiltonian and the MI. For such a purpose, we study the dependence of these expressions with the cross-ratio

$$\eta := \frac{(b_1 - a_1)(b_2 - a_2)}{(a_2 - a_1)(b_2 - b_1)} \in (0, 1), \quad (6.173)$$

which is the natural geometric parameter of the problem because of the conformal invariance of the model. The outcome of this subsection is that the function (6.165) can be expressed as in (6.182), in term of Hypergeometric functions, and the eigenvectors (6.167–6.170) can be expressed as in (6.184–6.187) and (6.191), which involve Appell functions.

Let us consider a general Möbius transformation $x \mapsto x' := f(x)$ given by

$$x' = f(x) := \frac{ax + b}{cx + d}, \quad (6.174)$$

where $a, b, c, d \in \mathbb{R}$ and $ad - cb > 0$. Such a transformation leaves the cross-ratio (6.173) invariant.

Let us first understand the dependence of the function $\alpha(s; a_1, b_1, a_2, b_2)$ with the interval endpoints. We can use (6.174) to make a change in the integration variables of the integrals (6.155), which are involved in the definition of the function α . After that, a straightforward computation shows that

$$\alpha(s; a_1, b_1, a_2, b_2) = \alpha(s; a'_1, b'_1, a'_2, b'_2), \quad (6.175)$$

where the two sets of intervals endpoints are related by

$$a'_j := f(a_j), \quad b'_j := f(b_j), \quad j = 1, 2. \quad (6.176)$$

Since (6.175) holds for any general Möbius transformation, we have that α depends on the intervals endpoints only through the cross-ratio (6.173), i.e. $\alpha = \alpha(s, \eta)$.

Similarly, a direct computation for the eigenfunctions shows the following covariance properties under the change of variable (6.174)

$$u_j(x'; \bar{q}') = e^{is\Omega(\bar{q})} u_j(x; \bar{q}), \quad (6.177)$$

$$v_j(x'; \bar{q}') = e^{is\Omega(\bar{q})} \frac{1}{f'(x)} u_j(x; \bar{q}), \quad (6.178)$$

where $u_j(x; \bar{q})$ and $v_i(x; \bar{q})$ are the eigenfunctions corresponding to the problem with endpoints $\bar{q} := (a_1, b_1, a_2, b_2)$ (idem for \bar{q}'), and the two sets of endpoints are related by (6.176). The real function $\Omega(\bar{q})$ is given by

$$\Omega(\bar{q}) := \frac{1}{2} \log \left(\frac{f'(a_1)f'(a_2)}{f'(b_1)f'(b_2)} \right). \quad (6.179)$$

Not surprisingly, the eigenfunctions $u_j(x)$ transforms as a scalar wave, whereas the eigenfunction $v_j(x)$ as their derivatives.

Simpler expressions are obtained when we specially take the Möbius transformation $x \mapsto x' := f_1(x)$ which sends the points $(a_1, b_1, a_2, b_2) \mapsto (0, \eta, 1, \infty)$, i.e.¹⁷

$$f_1(x) := \frac{(b_2 - a_2)(x - a_1)}{(a_2 - a_1)(b_2 - x)}, \quad (6.180)$$

$$f_1'(x) = \frac{(b_2 - a_1)(b_2 - a_2)}{(a_2 - a_1)(b_2 - x)^2} = \frac{1}{b_2 - a_1} \left(\frac{b_2 - a_2}{a_2 - a_1} + 2x' + \frac{a_2 - a_1}{b_2 - a_2} x'^2 \right). \quad (6.181)$$

With this transformation, we get from (6.165) the compact formula

$$\alpha(s, \eta) = -\frac{{}_2F_1(1 + is, -is; 1; \eta)}{{}_2F_1(1 - is, is; 1; \eta)}, \quad (6.182)$$

where ${}_2F_1(a, b; c; x)$ is the Gaussian or ordinary hypergeometric function. To derive this result, we used the integral representation of such a function

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1} (1-t)^{c-b-1} (1-tx)^{-a}, \quad (6.183)$$

which holds for $x < 1$ and $\text{Re}(c) > \text{Re}(b) > 0$.¹⁸ Expression (6.182) shows explicitly the dependence of α through the cross ratio, and the fact that it is a phase factor. For $s = 0$ we have $\alpha(0, \eta) = -1$, and it reaches a value dependent on η for $s \rightarrow \pm\infty$ (see below). We also have $\alpha(-s, \eta) = \alpha(s, \eta)^*$. For a fixed $s \neq 0$, we have that $\lim_{\eta \rightarrow 0} \alpha(s, \eta) = -1$ and $\lim_{\eta \rightarrow 1} \alpha(s, \eta) = 1$.

Applying the transformation (6.180) to the eigenvectors (6.167–6.170) and using the

¹⁷More carefully, we shall take the Möbius transformation which transforms $(a_1, b_1, a_2, b_2) \rightarrow (0, \eta, 1, \Lambda)$ with $\Lambda > 1$, and in the end, we take $\Lambda \rightarrow \infty$.

¹⁸When $\text{Re}(b) = 0$, which occurs in (6.182), equation (6.183) has to be understood as the f -regularization explained on (6.131).

expressions (6.177–6.178), we arrive to¹⁹

$$u_1(x) = \frac{1}{\sqrt{4\pi|s|}} e^{-is \log\left(\frac{x'(1-x')}{\eta-x'}\right)}, \quad (6.184)$$

$$v_1(x) = \sqrt{\frac{|s|}{4\pi}} f_1'(x) e^{-is \log\left(\frac{x'(1-x')}{\eta-x'}\right)} \left(\frac{1}{x'} + \frac{1}{x'-1} - \frac{1}{x'-\eta} \right), \quad (6.185)$$

$$\begin{aligned} u_2(x) = & i \frac{s}{\sqrt{4\pi|s|}} \left(\frac{x'}{\eta} \right)^{-is} \left[\frac{1}{is} F_1 \left(-is; is, -is; 1-is; x', \frac{x'}{\eta} \right) \right. \\ & + \frac{\alpha(s, \eta)}{1-is} \frac{x'}{\eta} F_1 \left(1-is; is, 1-is; 2-is; x', \frac{x'}{\eta} \right) \\ & \left. - \frac{x'}{1-is} F_1 \left(1-is; 1+is, -is; 2-is; x', \frac{x'}{\eta} \right) \right], \quad x \in A_1, \end{aligned} \quad (6.186)$$

$$v_2(x) = \sqrt{\frac{|s|}{4\pi}} f_1'(x) e^{-is \log\left(\frac{x'(1-x')}{\eta-x'}\right)} \left(\frac{1}{x'} + \frac{\alpha(s, \eta)}{x'-\eta} - \frac{1}{x'-1} \right), \quad (6.187)$$

where $F_1(a; \beta_1, \beta_1; c; z_1, z_2)$ in (6.186) is the Appell Hypergeometric function of two variables. Such a function has the following integral representation

$$F_1(a; \beta_1, \beta_1; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-tz_1)^{-\beta_1} (1-tz_2)^{-\beta_2}, \quad (6.188)$$

for $x, y < 1$ and $\text{Re}(a) > 0$ and $\text{Re}(c-a) > 0$.²⁰ We remark that formula (6.186) only holds for $x \in A_1$. In such a case, $x' \in (0, \eta)$, and therefore, the arguments of the Appell's functions belong to their domains of analyticity.

To get an expression for $u_2(x)$ valid for $x \in A_2$ in terms of Appell functions, we must consider a different Möbiüs transformation $x \mapsto \tilde{x} := f_2(x)$, which sends $(a_1, b_1, a_2, b_2) \mapsto (1, \infty, 0, \eta)$,

$$f_2(x) := \frac{(b_1 - a_1)(x - a_2)}{(a_2 - a_1)(x - b_1)}, \quad (6.189)$$

$$f_2'(x) = \frac{(b_1 - a_1)(a_2 - b_1)}{(a_2 - a_1)(x - b_1)^2} = \frac{1}{a_2 - b_1} \left(\frac{b_1 - a_1}{a_2 - a_1} + 2\tilde{x} + \frac{a_2 - a_1}{b_1 - a_1} \tilde{x}^2 \right). \quad (6.190)$$

¹⁹Here, we discard a global constant phase factor, which is the same for all the eigenvectors, and therefore, it does not modify the orthonormalization condition and the condition (6.108).

²⁰For $\text{Re}(a) = 0$, the integral (6.188) has to be understood in the same way we have explained in footnote 18.

Then, we obtain for $x \in A_2$

$$\begin{aligned}
u_2(x) = & \frac{-is}{\sqrt{4\pi|s|}} \left(\frac{\tilde{x}}{\eta}\right)^{-is} \left[\frac{1}{is} F_1 \left(-is; is, -is; 1 - is; \tilde{x}, \frac{\tilde{x}}{\eta} \right) \right. \\
& + \frac{\alpha(s, \eta)}{1 - is} \frac{\tilde{x}}{\eta} F_1 \left(1 - is; is, 1 - is; 2 - is; \tilde{x}, \frac{\tilde{x}}{\eta} \right) \\
& \left. - \frac{\tilde{x}}{1 - is} F_1 \left(1 - is; 1 + is, -is; 2 - is; \tilde{x}, \frac{\tilde{x}}{\eta} \right) \right], \tag{6.191}
\end{aligned}$$

which is the same expression valid for $u_2(x)$ in the first interval (up to a minus global sign), but evaluated in \tilde{x} instead of x' . This expression indicates that for any point $x_1 \in A_1$ exists a point $x_2 \in A_2$ such that $u_2(x_1) = -u_2(x_2)$, and viceversa. In the next subsection, we show that all the eigenvectors are classified according to such ‘‘parity symmetry’’.

6.3.2.3. Parity symmetry of the eigenvectors

The aim of this subsection is to study the behavior of the eigenfunctions under two different conformal transformations, The first one is the Möbius transformation that interchange the two intervals (eq. (6.192)), and the second one it the Möbius transformation that reflects each interval into itself. The outcome of this subsections is that the eigenfunctions satisfy the symmetry properties (6.199-6.202) and (6.204-6.207). These expressions will be used in subsection 6.3.2.5 to express the final result of the modular Hamiltonian in a more amenable way.

Let us first in introduce the Möbius transformation $x \mapsto \bar{x} := p(x)$ that interchange the two intervals $(a_1, b_1, a_2, b_2) \xrightarrow{p} (a_2, b_2, a_1, b_1)$, namely

$$\bar{x} = p(x) := \frac{a_1 a_2 (x - b_1 - b_2) - b_1 b_2 (x - a_1 - a_2)}{x(a_1 + a_2 - b_1 - b_2) + (b_1 b_2 - a_1 a_2)}, \tag{6.192}$$

$$p'(x) = \frac{(b_1 - a_1)(b_2 - a_1)(a_2 - b_1)(a_2 - b_2)}{[x(a_1 + a_2 - b_1 - b_2) + (b_1 b_2 - a_1 a_2)]^2} > 0, \tag{6.193}$$

where (6.192) is the same as (6.93), which indicates that \bar{x} is the conjugate point of the point x , i.e. $\omega(\bar{x}) = \omega(x)$. Specializing this transformation on the relations (6.177-6.178), we get

$$u_i(\bar{x}; a_2, b_2, a_1, b_1) = u_i(x; a_1, b_1, a_2, b_2), \tag{6.194}$$

$$v_i(\bar{x}; a_2, b_2, a_1, b_1) = \frac{1}{p'(x)} u_i(x; a_1, b_1, a_2, b_2), \tag{6.195}$$

where in this case we have that $\Omega(a_1, b_1, a_2, b_2) = 0$. On the other hand, from (6.167-6.169),

we easily see that

$$u_1(x; a_2, b_2, a_1, b_1) = u_1(x; a_1, b_1, a_2, b_2), \quad (6.196)$$

$$v_1(x; a_2, b_2, a_1, b_1) = v_1(x; a_1, b_1, a_2, b_2), \quad (6.197)$$

$$v_2(x; a_2, b_2, a_1, b_1) = -v_2(x; a_1, b_1, a_2, b_2). \quad (6.198)$$

Therefore, using additionally (6.186) and (6.191) for $u_2(x)$, we conclude we have the following parity symmetries

$$u_1(\bar{x}) = u_1(x), \quad (6.199)$$

$$v_1(\bar{x}) = \frac{1}{p'(x)} v_1(x), \quad (6.200)$$

$$u_2(\bar{x}) = -u_2(x), \quad (6.201)$$

$$v_2(\bar{x}) = -\frac{1}{p'(x)} v_2(x). \quad (6.202)$$

That means that the first set of eigenfunctions is even and the second one is odd under taking the conjugate point \bar{x} . We remark that in these expressions, the same eigenfunctions for the same endpoints (a_1, b_1, a_2, b_2) appear at both sides of the expressions.

Now, we consider the Möbius transformation $x \mapsto \hat{x} := q(x)$ that reflects each interval into itself, i.e. $(a_1, b_1, a_2, b_2) \xrightarrow{q} (b_1, a_1, b_2, a_2)$. It is given by

$$\hat{x} = q(x) := \frac{x(a_2 b_2 - a_1 b_1) - a_1 a_2 (b_2 - b_1) - b_1 b_2 (a_2 - a_1)}{x(b_2 + a_2 - b_1 - a_1) + a_1 b_1 - a_2 b_2}. \quad (6.203)$$

Under this transformation, the eigenvectors transform as

$$u_{1,-s}(\hat{x}) = e^{is\Omega(\bar{q})} u_{1,s}(x), \quad (6.204)$$

$$u_{2,-s}(\hat{x}) = e^{is\Omega(\bar{q})} (-1) \alpha(-s, \eta) u_{2,s}(x), \quad (6.205)$$

$$v_{1,-s}(\hat{x}) = e^{is\Omega(\bar{q})} \frac{(-1)}{q'(x)} v_{1,s}(x), \quad (6.206)$$

$$v_{2,-s}(\hat{x}) = e^{is\Omega(\bar{q})} \frac{\alpha(-s, \eta)}{q'(x)} v_{2,s}(x), \quad (6.207)$$

where now we explicitly write the dependence of the eigenfunctions with the parameter s , because the above expressions relate a eigenfunction of eigenvalue s with the one of eigenvalue $-s$. In this case, we have a no null phase factor $\Omega(\bar{q}) := 2 \log \left(\frac{b_2 - b_1}{a_2 - a_1} \right)$.

6.3.2.4. Mutual information

In this subsection, we compute the mutual information and the Rényi mutual informations, using the expressions (6.110) and (6.112) and the eigenvectors obtained in the

previous sections. The results are given by formulas (6.215-6.216) and (6.217-6.218), and showed in figures 6.2, 6.3, and 6.4.

According to (3.27), the MI can be expressed as

$$I(A_1, A_2) = S(A_1) + S(A_2) - S(A_1 \cup A_2), \quad (6.208)$$

where the one interval entropies $S(A_j)$ are obtained from (6.142) and the two-interval EE follows from (6.110) and (6.112). This last is given by

$$S(A) = \sum_{k=1}^2 \int_0^{+\infty} ds \int_{A^{(\epsilon)}} dx g(s) u_{k,s}(x) v_{k,s}(x)^*, \quad (6.209)$$

where $A^{(\epsilon)} := A_1^{(\epsilon)} \cup A_2^{(\epsilon)}$ with $A_j^{(\epsilon)} := (a_j + \epsilon, b_j - \epsilon)$ is the regularized region. It is important to emphasize that the MI is finite and independent of the regularization. However, each term on the r.h.s. of equation (6.208) is UV-divergent, and their corresponding regularizations cannot be chosen independently. They correspond to evaluating the integrals (6.142) and (6.209) along the regularized regions, as we have already explain, with the same cutoff parameter $\epsilon > 0$ for all the terms. In the end, we take the limit $\epsilon \rightarrow 0^+$ and we get the finite desired result for the MI.

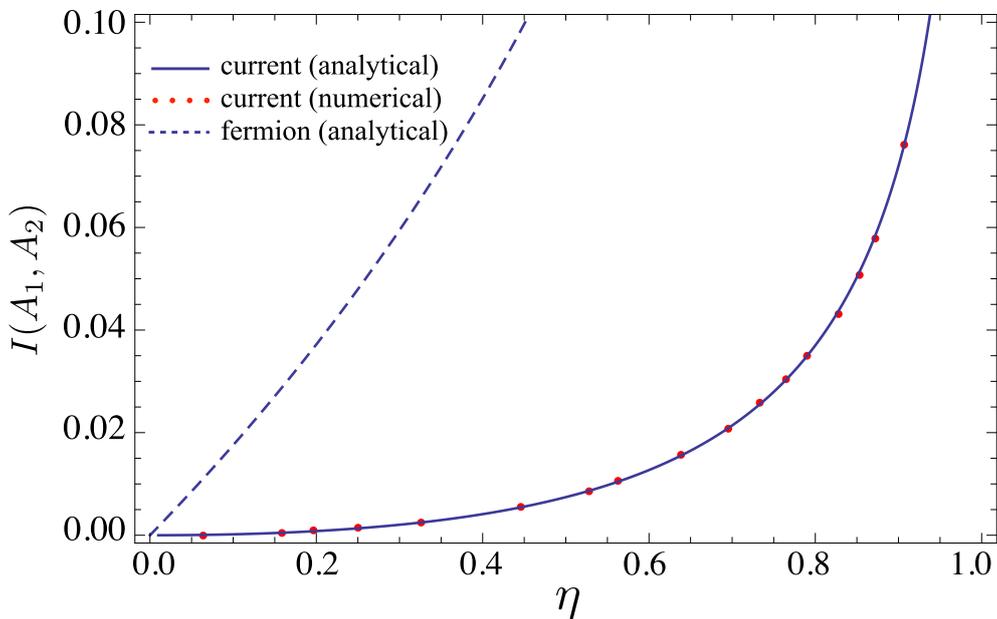


Figure 6.2: The mutual information $I(A_1, A_2)$ as function of the cross ratio η . The continuous solid line corresponds to the mutual information obtained by numerical integration of the analytic expression (6.215) and (6.216). The red points correspond to the simulation on the lattice, which is explained in section 6.3.3. The dashed line is the free chiral fermion mutual information $-\frac{1}{6} \log(1-\eta)$.

It is convenient to express the two-interval entropy as $S(A) := S_1(A) + S_2(A)$, where $S_k(A)$ denotes the term in (6.209) involving the functions $u_{k,s}(x)$ and $v_{k,s}(x)$ ($k = 1, 2$).

Using the formulas (6.167-6.168) for the eigenvectors, we easily obtain

$$\begin{aligned}
S_1(A) &= \int_0^{+\infty} ds \int_{A(\epsilon)} dx g(s) u_{1,s}(x) v_{1,s}(x)^* = \frac{1}{24} \int_{A(\epsilon)} dx \omega'(x) \\
&= \frac{1}{12} \log \left(\frac{(a_2 - b_1)(b_2 - a_1)}{(b_2 - b_1)(a_2 - a_1)} \right) + \frac{1}{12} \log \left(\frac{b_1 - a_1}{\epsilon} \right) + \frac{1}{12} \log \left(\frac{b_2 - a_2}{\epsilon} \right) \\
&= \frac{1}{12} \log(1 - \eta) + \frac{1}{2} S(A_1) + \frac{1}{2} S(A_2). \tag{6.210}
\end{aligned}$$

Direct treatment of the integral for $S_2(A)$ is more complicated due to the presence of the hypergeometric and Appell functions in $u_2(x)$. We find convenient to use the following trick. Since the integral in (6.209) is regularized, keeping ϵ fixed, we can replace

$$\begin{aligned}
\int_{A(\epsilon)} dx u_{2,s}(x) v_{2,s}(x)^* &= \lim_{\delta s \rightarrow 0} \int_{A(\epsilon)} dx u_{2,s}(x) v_{2,s+\delta s}(x)^* \\
&= \lim_{\delta s \rightarrow 0} \left[\int_A dx u_{2,s}(x) v_{2,s+\delta s}(x)^* - \sum_{j=1}^2 \left(\int_{a_j}^{a_j+\epsilon} dx u_{2,s}(x) v_{2,s+\delta s}(x)^* + \int_{b_j-\epsilon}^{b_j} dx u_{2,s}(x) v_{2,s+\delta s}(x)^* \right) \right] \\
&= - \lim_{\delta s \rightarrow 0} \sum_{j=1}^2 \left(\int_{a_j}^{a_j+\epsilon} dx u_{2,s}(x) v_{2,s+\delta s}(x)^* + \int_{b_j-\epsilon}^{b_j} dx u_{2,s}(x) v_{2,s+\delta s}(x)^* \right), \tag{6.211}
\end{aligned}$$

where in the last step we have used the fact that vectors $u_s(x)$ and $v_{s+\delta s}(x)$ are orthogonal for $\delta s \neq 0$. The advantage of doing that is that we do not need now the precise behavior of the eigenfunctions along the intervals but only in a small region near the endpoint of the intervals. Then, we can just take the leading terms of $u_s(x)$ and $v_{s+\delta s}(x)$ since all other subleading terms in ϵ will disappear in the limit $\epsilon \rightarrow 0^+$. From (6.167–6.170), these leading terms are

$$u_{2,s}(x) \sim \begin{cases} \frac{(-1)^{j+1}}{\sqrt{4\pi|s|}} e^{-is\omega(x)} & \text{for } x \rightarrow a_j, \\ \frac{(-1)^j}{\sqrt{4\pi|s|}} \alpha(s, \eta) e^{-is\omega(x)} & \text{for } x \rightarrow b_j, \end{cases} \tag{6.212}$$

$$v_{2,s}(x) \sim \begin{cases} (-1)^{j+1} \sqrt{\frac{|s|}{4\pi}} \frac{e^{-is\omega(x)}}{x-a_j} & \text{for } x \rightarrow a_j, \\ (-1)^{j+1} \alpha(s, \eta) \sqrt{\frac{|s|}{4\pi}} \frac{e^{-is\omega(x)}}{x-b_j} & \text{for } x \rightarrow b_j, \end{cases} \tag{6.213}$$

Replacing these expressions into (6.211), we get

$$S_2(A) = \frac{1}{12} \log(1 - \eta) + \frac{1}{2} S(A_1) + \frac{1}{2} S(A_2) - \int_0^{+\infty} ds \frac{g(s)}{2\pi} i \alpha(s, \eta) \partial_s \alpha^*(s, \eta). \tag{6.214}$$

Taking into account that $-i \alpha(s, \eta) \partial_s \alpha^*(s, \eta) = i \partial_s \log(\alpha(s, \eta))$ and integrating by parts the last term in (6.214), we finally obtain

$$I(A_1, A_2) = -\frac{1}{6} \log(1 - \eta) + U(\eta), \quad (6.215)$$

$$U(\eta) = -\frac{i\pi}{2} \int_0^{+\infty} ds \frac{s}{\sinh^2(\pi s)} \log \left(\frac{{}_2F_1(1 + is, -is; 1; \eta)}{{}_2F_1(1 - is, is; 1; \eta)} \right). \quad (6.216)$$

The first term in (6.215) coincides with the mutual information of the free chiral fermion field [36, 133]. We could not express the integral in (6.216) in terms of standard known functions, and hence, it has to be computed numerically. The result for $U(\eta)$ is always negative, as it must be, considering that the free chiral current is a subnet of the free chiral fermion algebra, and hence, its MI has to be smaller. In figure 6.2, we show a plot of the MI, while the function $U(\eta)$ is shown in figure 6.3.

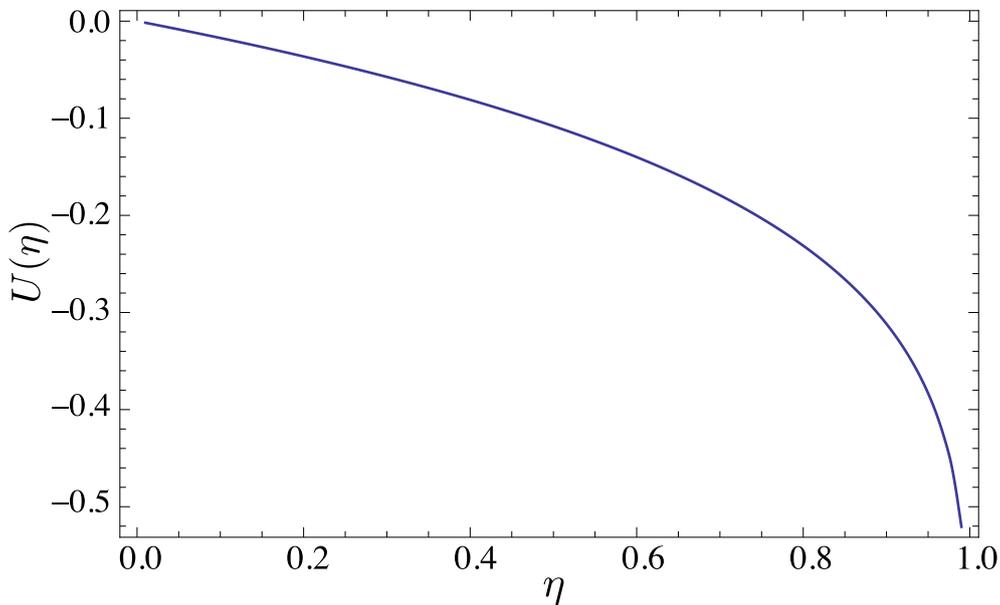


Figure 6.3: The function $U(\eta)$ as a function of the cross-ratio η . This function is negative because the free chiral current is a subnet of the free chiral fermion, which it has $U(\eta) = 0$. It does not have the $\eta \leftrightarrow (1 - \eta)$ symmetry expected for the case where the entropy of a two-interval region is equal to the entropy of its complement. $U(\eta) \sim -\frac{1}{2} \log(-\log(1 - \eta))$ for $\eta \rightarrow 1$.

The Rényi mutual information $I_\alpha(\eta) = S_\alpha(A_1) + S_\alpha(A_2) - S_\alpha(A_1 \cup A_2)$ can be computed in the same fashion, using (6.111) and (6.113). A straightforward computation gives²¹

$$I_\alpha(\eta) = -\frac{1 + \alpha}{12\alpha} \log(1 - \eta) + U_\alpha(\eta), \quad (6.217)$$

$$U_\alpha(\eta) = \frac{i\alpha}{2(\alpha - 1)} \int_0^{+\infty} ds (\coth(\pi s \alpha) - \coth(\pi s)) \log \left(\frac{{}_2F_1(1 + is, -is; 1; \eta)}{{}_2F_1(1 - is, is; 1; \eta)} \right). \quad (6.218)$$

In figure 6.4, we display $I_\alpha(\eta)$ for some values of α .

²¹Do not confuse the α -index of the Rényi MI with the function $\alpha(s, \eta)$ defined above in equation (6.182).

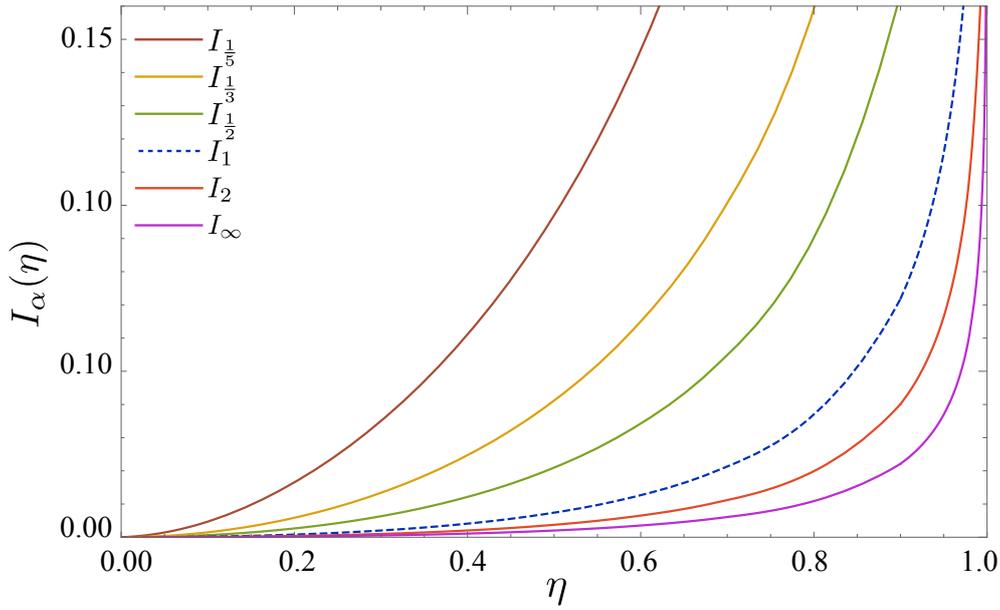


Figure 6.4: The Rényi mutual information $I_\alpha(\eta)$ as a function of the cross-ratio η for different values of α .

Asymptotic behavior of the mutual information

Now, we study the asymptotic behavior of the MI for large and short distances between the intervals. The $\eta \rightarrow 0^+$ limit corresponds to the large distance limit between the intervals. Since the integrand of (6.216) is analytic at $\eta = 0$, a simple Taylor expansion reveals the following asymptotic behavior

$$U(\eta) = -\frac{1}{6}\eta - \frac{1}{15}\eta^2 - \frac{13}{315}\eta^3 + \mathcal{O}(\eta^4), \quad \eta \rightarrow 0^+, \quad (6.219)$$

This gives

$$I(\eta) \sim \frac{\eta^2}{60} + \frac{\eta^3}{70} + \mathcal{O}(\eta^4), \quad \eta \rightarrow 0^+. \quad (6.220)$$

The first term coincides with the general result for the leading term of the large distance expansion of the mutual information for a CFT [122, 123, 134, 135]. For two intervals, this is given by

$$I(\eta) \sim \frac{\sqrt{\pi}\Gamma(2\Delta + 1)}{4^{2\Delta+1}\Gamma(2\Delta + 3/2)}\eta^{2\Delta}, \quad \eta \rightarrow 0^+, \quad (6.221)$$

where Δ is the lowest dimensional operator of the theory. In the present model, this is $J(x) = \partial\phi(x)$ and has $\Delta = 1$. The fermion has $\Delta = 1/2$ and a different behavior $I(\eta) \sim \frac{1}{6}\eta$ at large distances, which is quite visible in figure 6.2.

The short distance limit $\eta \rightarrow 1^-$ is more tricky, since the integrand of (6.216) converges to zero in a non-uniform way in such a limit. The main contribution to $U(\eta)$ in this limit comes from $s \sim 0$. We have to expand the Hypergeometric functions at the numerator and

denominator inside the logarithm in (6.216) for $\eta \rightarrow 1^-$ and $s \sim 0$ to get

$$U(\eta) \sim -\frac{i\pi}{2} \int_0^{+\infty} ds \frac{s}{\sinh^2(\pi s)} \log \left(\frac{i - s \log(1 - \eta)}{i + s \log(1 - \eta)} \right) \sim -\frac{1}{2} \log(-\log(1 - \eta)). \quad (6.222)$$

This gives the expansion

$$I(\eta) \sim -\frac{1}{6} \log(1 - \eta) - \frac{1}{2} \log(-\log(1 - \eta)), \quad \eta \rightarrow 1. \quad (6.223)$$

6.3.2.5. Modular Hamiltonian

Now, we have now all the necessary elements to compute the modular Hamiltonian. It is given by

$$K_A = \int_{A \times A} dx dy J(x) \mathcal{K}(x, y) J(y), \quad (6.224)$$

where, according to (6.140), the kernel is

$$\mathcal{K}(x, y) = \sum_{k=1}^2 \int_{-\infty}^{+\infty} ds u_{k,s}(x) \pi |s| u_{k,s}(y)^*. \quad (6.225)$$

This kernel is real and symmetric.

Because the expression for the modular Hamiltonian turns out to be quite complex, we start with some preliminaries about how we are going to express the results. Our first simplification will be to express the kernel in the four regions $A_1 \times A_1$, $A_1 \times A_2$, $A_2 \times A_1$ and $A_2 \times A_2$, called respectively \mathcal{K}_{11} , \mathcal{K}_{12} , \mathcal{K}_{21} and \mathcal{K}_{22} , in terms of the kernel in the first interval alone $A_1 \times A_1$, using the parity symmetry of the eigenfunctions developed in section 6.3.2.3.

Let us start by computing the contribution of the first eigenvector u_1 to \mathcal{K}_{11} ,

$$\begin{aligned} \int_{-\infty}^{+\infty} ds u_{1,s}(x) \pi |s| u_{1,s}(y)^* &= \frac{1}{4} \int_{-\infty}^{+\infty} ds s e^{is(\omega(x) - \omega(y))} \\ &= \frac{\pi}{2} \frac{\delta(x - x_1(\omega(y)))}{\omega'(x)} = \frac{\pi}{2} \omega'(x)^{-1} \delta(x - y), \end{aligned} \quad (6.226)$$

where we set $x, y \in A_1$, and hence, we have summed over only one of the roots of $\omega(x) = \omega(y)$ in the delta function. We will find convenient to write

$$\mathcal{K}_{11}(x, y) = \pi \omega'(x)^{-1} \delta(x - y) + \mathcal{N}(x, y), \quad x, y \in A_1, \quad (6.227)$$

that is, we have doubled the delta function contribution from $u_1(x)$ and the remaining

part comes from (6.225) and (6.226)

$$\mathcal{N}(x, y) := \int_{-\infty}^{+\infty} ds \pi |s| [u_2(x)u_2(y)^* - u_1(x)u_1(y)^*], \quad x, y \in A_1. \quad (6.228)$$

It turns out that $\mathcal{N}(x, y)$ is a regular distribution.²² Hence, it gives a purely non-local contribution to the modular Hamiltonian.

Now, we recall the parity symmetry of the eigenfunctions studied in section 6.3.2.3 under the conformal transformation $x \mapsto \bar{x} = p(x)$ that interchanges the intervals (eq. (6.192)). We have that $u_1(\bar{x}) = u_1(x)$ and $u_2(\bar{x}) = -u_2(x)$. These relations give the following relations between the kernel of the modular Hamiltonian in the different sectors

$$\mathcal{K}_{12}(x, y) = \mathcal{K}_{21}(y, x) = -\mathcal{N}(x, \bar{y}), \quad x \in A_1, y \in A_2, \quad (6.229)$$

$$\mathcal{K}_{22}(x, y) = \mathcal{K}_{11}(\bar{x}, \bar{y}), \quad x \in A_2, y \in A_2. \quad (6.230)$$

These relations, together with (6.227), reduce the problem to the one of finding the form of the kernel $\mathcal{N}(x, y)$ in $A_1 \times A_1$. Unless otherwise is stated, in the following we will assume that $x, y \in A_1$.

A different parity symmetry (eq. (6.203)) implies that the kernel $\mathcal{N}(x, y)$ satisfies

$$\mathcal{N}(x, y) = \mathcal{N}(\hat{x}, \hat{y}), \quad x, y \in A_1, \quad (6.231)$$

where the transformation $x \mapsto \hat{x} = q(x)$ is given by (6.203). This follows from the corresponding symmetry of the eigenvectors (6.204) and (6.205).

Another simplification is that we can relate all two-interval cases with cross-ratio η , between the four endpoints of the intervals, to the case where the two-interval region is the standard one $A_\eta := (0, \eta) \cup (1, \infty)$. This is done applying the conformal transformation $x \mapsto x' = f_1(x)$, given by (6.174), on the eigenvectors, eqs. (6.177-6.178). This simply gives

$$\mathcal{N}_A(x, y) = \mathcal{N}_\eta(x', y'), \quad (6.232)$$

where we wrote explicitly the dependence on the two-interval regions. In the following, we will call simply $\mathcal{N}(x', y')$ to the kernel on the r.h.s. of (6.232), and keep $x', y' \in (0, \eta)$. That is, we focus on the first interval $(0, \eta)$ in the case where the region is A_η .

An evaluation of $\mathcal{N}(x', y')$ requires the integration over s , which turns out to be the Fourier transform of products of Appell functions contained in $u_2(x)$, eq. (6.186). This obscures the structure of the kernel due to the complexity of these functions, and in particular, the analysis of the possible singular terms. Instead, we proceed in the following

²²It is given by a locally integrable function.

way. We first write the vectors $u_s(x)$ as

$$u_s(x') = -i \operatorname{sign}(s) \int_0^{x'} d\tilde{x} v_s(\tilde{x}). \quad (6.233)$$

Then, we make the integral in s more amenable writing

$$\mathcal{N}(x', y') = \int_0^{x'} d\tilde{x} \int_0^{y'} d\tilde{y} \mathcal{M}(\tilde{x}, \tilde{y}), \quad (6.234)$$

where

$$\mathcal{M}(\tilde{x}, \tilde{y}) := \int_{-\infty}^{+\infty} ds \pi |s| [v_2(\tilde{x}) v_2^*(\tilde{y}) - v_1(\tilde{x}) v_1^*(\tilde{y})]. \quad (6.235)$$

We split the above kernel into two contributions

$$\mathcal{M}(x', y') =: \mathcal{M}_1(x', y') + \mathcal{M}_2(x', y'), \quad (6.236)$$

corresponding to $v_1(x)$ and $v_2(x)$. Using (6.185), we get

$$\mathcal{M}_1(x', y') = \frac{\pi}{2} \tilde{\omega}'(x') \tilde{\omega}'(y') \delta''(\tilde{\omega}(x') - \tilde{\omega}(y')), \quad (6.237)$$

where

$$\tilde{\omega}(x') := \log\left(\frac{x'(1-x')}{\eta-x'}\right). \quad (6.238)$$

For the other term we use (6.187), and hence we get

$$\begin{aligned} \mathcal{M}_2(x', y') &= \int_{-\infty}^{+\infty} ds \pi |s| \frac{1}{f_1'(x)} \frac{1}{f_1'(y)} v_{2,s}(x) v_{2,s}^*(y) \\ &= \frac{1}{4} \int_{-\infty}^{+\infty} ds s^2 e^{-is(\tilde{\omega}(x') - \tilde{\omega}(y'))} \left(\frac{1}{x'} + \frac{\alpha(s, \eta)}{x' - \eta} - \frac{1}{x' - 1}\right) \left(\frac{1}{y'} + \frac{\alpha(-s, \eta)}{y' - \eta} - \frac{1}{y' - 1}\right) \\ &=: \mathcal{M}_{2,\phi}(x', y') + \mathcal{M}_{2,\alpha}(x', y'), \end{aligned} \quad (6.239)$$

where

$$\mathcal{M}_{2,\phi}(x', y') := -\frac{\pi}{2} \left[\left(\frac{1}{x'} - \frac{1}{x' - 1}\right) \left(\frac{1}{y'} - \frac{1}{y' - 1}\right) + \frac{1}{x' - \eta} \frac{1}{y' - \eta} \right] \delta''(z), \quad (6.240)$$

$$\mathcal{M}_{2,\alpha}(x', y') := \frac{\pi}{2} \left(\frac{1}{x'} - \frac{1}{x' - 1}\right) \frac{1}{y' - \eta} \hat{\alpha}(z) + \frac{\pi}{2} \frac{1}{x' - \eta} \left(\frac{1}{y'} - \frac{1}{y' - 1}\right) \hat{\alpha}(-z), \quad (6.241)$$

where $z := \tilde{\omega}(x') - \tilde{\omega}(y')$ and we have also introduced the function

$$\hat{\alpha}(z, \eta) := \frac{1}{2\pi} \int_{-\infty}^{\infty} ds s^2 \alpha(s, \eta) e^{isz}. \quad (6.242)$$

(6.240) and (6.241) are respectively the α -independent and α -dependent contributions to

the kernel $\mathcal{M}_2(x', y')$.

This gives the kernel $\mathcal{N}(x', y')$ as a double integral over the sum of (6.237), (6.240), and (6.241). The final result depends on the Fourier transform of the function $s^2 \alpha(s, \eta)$, which has to be computed numerically. This numerical computation can be done after we have extracted the leading terms for $s \rightarrow \infty$ from $\alpha(s, \eta)$. This will also help understanding the structure of singularities of these kernels. In the following, we will make a further analysis of their local and non-local parts.

Structure of singular terms

The asymptotic behavior of the hypergeometric functions in $\alpha(s, \eta)$ for large s can be computed using the integral representation (6.183) and the saddle point approximation. This is straightforward. The leading term was computed for example in [136]. Extending this calculation to include fluctuations around the saddle point, we get the asymptotic expansion

$$\alpha(s, \eta) = \alpha_0 + \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \frac{\alpha_3}{s^3} + \mathcal{O}(|s|^{-4}), \quad |s| \rightarrow \infty, \quad (6.243)$$

where

$$\alpha_0 = (2\eta - 1) + i 2\sqrt{\eta(1-\eta)} \operatorname{sign}(s), \quad (6.244)$$

$$\alpha_1 = \frac{i}{2} (2\eta - 1) - \sqrt{\eta(1-\eta)} \operatorname{sign}(s), \quad (6.245)$$

$$\alpha_2 = -\frac{i}{16} \frac{1}{\sqrt{\eta(1-\eta)}} \operatorname{sign}(s), \quad (6.246)$$

$$\alpha_3 = -\frac{1}{32} \frac{1}{\sqrt{\eta(1-\eta)}} \operatorname{sign}(s) + i \frac{1}{32} \frac{(2\eta - 1)}{\eta(1-\eta)}. \quad (6.247)$$

Instead of extracting these asymptotic terms directly, we write

$$\alpha(s, \eta) = \tilde{\alpha}_0 + \frac{\tilde{\alpha}_1}{s} + \frac{\tilde{\alpha}_2}{s^2} + \frac{\tilde{\alpha}_3}{s^3} + \alpha_r(s, \eta), \quad (6.248)$$

where now

$$\tilde{\alpha}_0 = (2\eta - 1) + i 2\sqrt{\eta(1-\eta)} \tanh\left(\frac{\pi}{2}s\right), \quad (6.249)$$

$$\tilde{\alpha}_1 = \frac{i}{2} (2\eta - 1) - \sqrt{\eta(1-\eta)} \tanh\left(\frac{\pi}{2}s\right), \quad (6.250)$$

$$\tilde{\alpha}_2 = -\frac{i}{16} \frac{1}{\sqrt{\eta(1-\eta)}} \tanh\left(\frac{\pi}{2}s\right), \quad (6.251)$$

$$\tilde{\alpha}_3 = -\frac{1}{32} \frac{1}{\sqrt{\eta(1-\eta)}} \tanh\left(\frac{\pi}{2}s\right) + i \frac{1}{32} \frac{(2\eta - 1)}{\eta(1-\eta)} \frac{s^2}{s^2 + 1}, \quad (6.252)$$

and the remainder function $\alpha_r(s, \eta)$ is smooth in the parameter s and $\alpha_r(s, \eta) \sim \frac{1}{s^4}$ when $|s| \rightarrow \infty$. The Fourier transform of $s^2\alpha(s, \eta)$ is

$$\begin{aligned} \hat{\alpha}(z, \eta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} ds s^2 \alpha(s, \eta) e^{isz} = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds s^2 \left[\tilde{\alpha}_0 + \frac{\tilde{\alpha}_1}{s} + \frac{\tilde{\alpha}_2}{s^2} + \frac{\tilde{\alpha}_3}{s^3} + \alpha_r(s, \eta) \right] e^{isz} \\ &= (1 - 2\eta) \delta''(z) + \frac{1}{2} (2\eta - 1) \delta'(z) + \frac{1}{\pi} \sqrt{\eta(1 - \eta)} (3 + \cosh(2z)) \operatorname{csch}^3(z) \\ &\quad + \frac{1}{2\pi} \sqrt{\eta(1 - \eta)} \sinh(2z) \operatorname{csch}^3(z) + \frac{1}{16\pi} \frac{1}{\sqrt{\eta(1 - \eta)}} \operatorname{csch}(z) \\ &\quad + \frac{1}{32\pi} \frac{1}{\sqrt{\eta(1 - \eta)}} \log\left(\tanh\left|\frac{z}{2}\right|\right) + \frac{1}{64} \frac{1 - 2\eta}{\eta(1 - \eta)} \operatorname{sign}(z) e^{-|z|} + \hat{\alpha}_r(z, \eta), \end{aligned} \quad (6.253)$$

where $\hat{\alpha}_r(z, \eta)$ is the Fourier transform of $s^2\alpha_r(s, \eta)$. This is a continuous function vanishing exponentially fast at infinity. This real function is computed numerically, and it is shown in figure (6.5) for some values of η . Putting all together, we finally get

$$\begin{aligned} \mathcal{M}(x', y') &= \mathcal{M}_1(x', y') + \mathcal{M}_2(x', y') = \mathcal{M}_1(x', y') + \mathcal{M}_{2,\alpha}(x', y') + \mathcal{M}_{2,\phi}(x', y') \\ &= m_{\delta''}(x', y'; \eta) \delta''(z) + m_{\delta'}(x', y'; \eta) \delta'(z) + m_i(x', y'; \eta) + m_r(x', y'; \eta), \end{aligned} \quad (6.254)$$

where

$$\begin{aligned} m_{\delta''}(x', y'; \eta) &:= \frac{\pi}{2} \tilde{\omega}'(x') \tilde{\omega}'(y') - \frac{\pi}{2} \left[\frac{1}{x'(1 - x')y'(1 - y')} + \frac{1}{(\eta - x')(\eta - y')} \right] \\ &\quad + \frac{\pi}{2} (2\eta - 1) \left[\frac{1}{r(x', y')} + \frac{1}{r(y', x')} \right], \end{aligned} \quad (6.255)$$

$$m_{\delta'}(x', y'; \eta) := -\frac{\pi}{4} (2\eta - 1) \left[\frac{1}{r(x', y')} - \frac{1}{r(y', x')} \right], \quad (6.256)$$

$$\begin{aligned} m_i(x', y'; \eta) &:= \sqrt{\eta(1 - \eta)} \frac{[r(x', y') - r(y', x')]^2}{[r(x', y') + r(y', x')]^3} \\ &\quad + 2\sqrt{\eta(1 - \eta)} \frac{r(x', y') r(y', x')}{[r(x', y') - r(y', x')]^2 [r(x', y') + r(y', x')]} \\ &\quad + \frac{1}{16} \frac{1}{\sqrt{\eta(1 - \eta)}} \frac{1}{r(x', y') + r(y', x')} \\ &\quad - \frac{1}{64} \frac{1}{\sqrt{\eta(1 - \eta)}} \frac{r(x', y') + r(y', x')}{r(x', y') r(y', x')} \log \left| \frac{r(x', y') - r(y', x')}{r(x', y') + r(y', x')} \right| \\ &\quad - \frac{\pi}{128} \frac{(2\eta - 1)}{\eta(1 - \eta)} [r(x', y') - r(y', x')] \left[\frac{\Theta(x - y)}{r(x', y')^2} - \frac{\Theta(y - x)}{r(y', x')^2} \right], \end{aligned} \quad (6.257)$$

$$m_r(x', y'; \eta) := -\frac{\pi}{2} \frac{1}{r(x', y')} \hat{\alpha}_r(z, \eta) - \frac{\pi}{2} \frac{1}{r(y', x')} \hat{\alpha}_r(-z, \eta), \quad (6.258)$$

and we have defined the positive polynomial function

$$r(x', y') := x'(1 - x')(\eta - y') > 0. \quad (6.259)$$

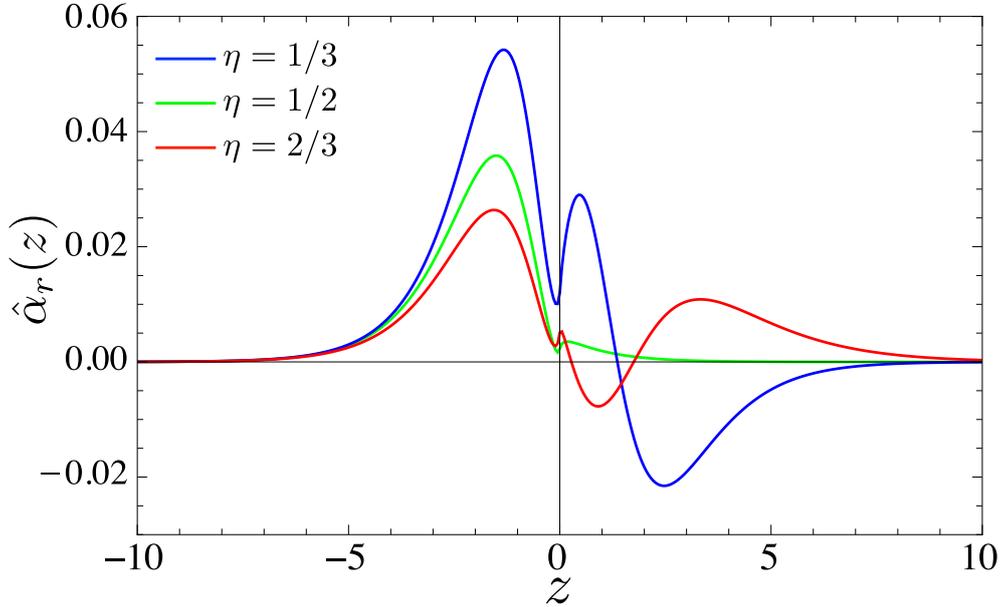


Figure 6.5: The function $\hat{\alpha}_r(z, \eta)$ for different values of η .

The singular terms can be simplified rewriting the Dirac delta distributions in terms of the variable $x - y$. A careful computation reveals the relations²³

$$m_{\delta''}(x', y'; \eta) \delta''(\tilde{\omega}(x') - \tilde{\omega}(y')) = 2\pi \eta (1 - \eta) \frac{x'(1 - x')(\eta - x')}{(\eta + x'^2 - 2\eta x')^3} \delta(x' - y'), \quad (6.260)$$

$$m_{\delta'}(x', y'; \eta) \delta'(\tilde{\omega}(x') - \tilde{\omega}(y')) = \frac{\pi}{4} \frac{(1 - 2\eta)}{\eta + x'^2 - 2\eta x'} \delta(x' - y'). \quad (6.261)$$

Therefore, upon integration in (6.234), these terms behave as ordinary functions, and they produce a singularity in $\mathcal{N}(x', y')$ which is just a jump in the first derivative for $x' = y'$.

In a similar way, expanding for $x' \sim y'$, the term (6.257) reveals that it behaves as $(x' - y')^{-2}$ for $x' \rightarrow y'$. Upon integration, this gives a $\log|x' - y'|$ singularity for $\mathcal{N}(x', y')$. Hence, this shows that the non-local kernel $\mathcal{N}(x', y')$ is given by a regular distribution.

The behavior of the kernel $\mathcal{N}(x', y')$ on the extremes of the interval (for example for $x' \rightarrow 0$ and y' fixed) can be seen more clearly from (6.237), (6.240), and (6.241). The only non-local contribution comes from the Fourier transform of $s^2\alpha(\eta, s)$,

$$\mathcal{M}(x', y') \sim \frac{\pi}{2x'y' - \eta} \hat{\alpha}(\log(x'), \eta), \quad x' \rightarrow 0. \quad (6.262)$$

²³The reason for a term containing two derivatives of the Dirac delta function becomes just a term proportional to $\delta(x' - y')$ is that the function $k_{\delta''}(x', y'; \eta)$ behaves as $(x' - y')^2$ for $x' \rightarrow y'$.

Since $\alpha(\eta, s)$ is an infinite differentiable function of s , its Fourier transform fall faster than any power of the variable z . Therefore $\mathcal{M}(x', y')$ is integrable on the boundary and that is the reason we have not needed to use a regularization in (6.234).²⁴ As a result, $\mathcal{N}(x', y')$ falls to zero faster than any power of $\log(x')^{-1}$ for $x' \rightarrow 0$.

A simplification in the structure of the non local kernel $\mathcal{N}(x', y')$ arises if we take into account that the integration $\int_{A'_1} dx' v_s^k(x') = 0$. Hence, using (6.235) we could write (6.234) as

$$\mathcal{N}(x', y') = - \int_0^{x'} d\tilde{x} \int_{y'}^{\eta} d\tilde{y} (m_i(\tilde{x}, \tilde{y}) + m_r(\tilde{x}, \tilde{y})), \quad x' < y', \quad (6.263)$$

$$\mathcal{N}(x', y') = - \int_{x'}^{\eta} d\tilde{x} \int_0^{y'} d\tilde{y} (m_i(\tilde{x}, \tilde{y}) + m_r(\tilde{x}, \tilde{y})), \quad x' > y'. \quad (6.264)$$

In this way, we avoid crossing the $\tilde{x} = \tilde{y}$ line in the integration, and therefore the delta functions (6.260) and (6.261) do not contribute. Moreover, the integrals are now completely regular and can be done numerically since we do not have to cross the singular points of the distributions. We checked these expressions coincide with (6.234).²⁵ A contour plot of $\mathcal{N}(x', y')$ for $\eta = 9/10$ is shown in figure 6.6.

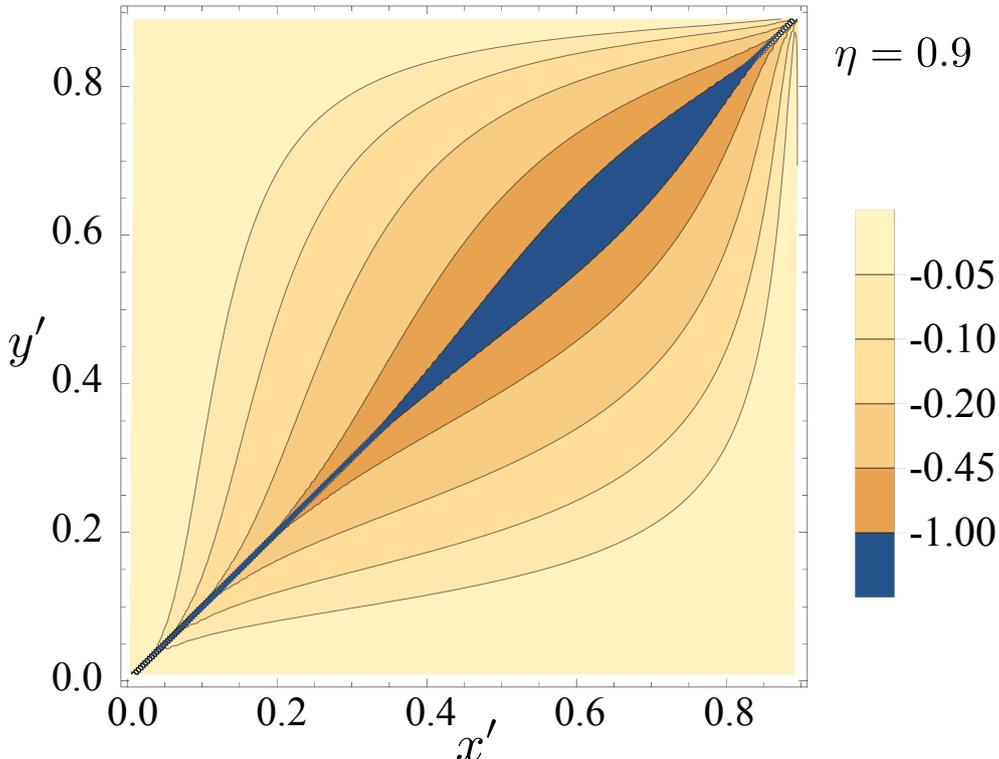


Figure 6.6: A contour plot of the kernel $\mathcal{N}(x', y')$ giving the non-local part of the modular Hamiltonian for two intervals for $\eta = 9/10$. $\mathcal{N}(x', y')$ has a logarithmic divergence along the diagonal.

²⁴That is, the regularizing terms in the integral of $v_s(x)$ as in (6.131) do not contribute if we make the integral in s first.

²⁵We have evaluated (6.234) extracting the singular contribution and integrating it analytically, and adding the numerical integral of the regular parts.

Summary of the modular Hamiltonian

Summarizing the results, the modular Hamiltonian contains a local part and a non-local part

$$K = K_{\text{loc}} + K_{\text{noLoc}}. \quad (6.265)$$

The local part, generated by the delta function in (6.227) and (6.230), gives a contribution to the modular Hamiltonian that writes on the full region A as

$$K_{\text{loc}} = \int_A dx \pi (\omega'(x))^{-1} : J(x)^2 := \int_A dx 2\pi (\omega'(x))^{-1} T(x), \quad (6.266)$$

where $T(x) = \frac{1}{2} : J^2(x) :$ is the energy density. The quantity $\beta(x) := 2\pi (\omega'(x))^{-1}$ acts as the local inverse temperature multiplying the energy density operator, and it dominates the limit of relative entropy between the vacuum and energetic localized excitations around x [37]. This term, written in terms of the energy density, is equal to the local term of the modular Hamiltonian for the free chiral fermion studied in section 6.2. This result coincides with general expectations for this term to be universal across two dimensional theories [37, 38].

The non-local part of the modular Hamiltonian is given by

$$\begin{aligned} K_{\text{noLoc}} = & \int_{A_1 \times A_1} dx dy J(x) \mathcal{N}(x, y) J(y) - \int_{A_1 \times A_2} dx dy J(x) \mathcal{N}(x, \bar{y}) J(y) \\ & - \int_{A_2 \times A_1} dx dy J(x) \mathcal{N}(\bar{x}, y) J(y) + \int_{A_2 \times A_2} dx dy J(x) \mathcal{N}(\bar{x}, \bar{y}) J(y). \end{aligned} \quad (6.267)$$

The relevant kernel $\mathcal{N}(x, y)$ follows from (6.263), (6.264), (6.257), and (6.258). In contrast to the case of the free chiral fermion, $\mathcal{N}(x, y)$ is less singular than the local term. It is given by an integrable function, having at most a $\log|x - y|$ singularity for $x \sim y$. Again in contrast to the fermion case, the modular Hamiltonian is completely non local, i.e. the kernel does not vanish identically in any open set of $A \times A$.

Modular flow

The modular flow acts over an operator $O \in \mathcal{A}(A)$ as the unitary transformation $O(t) := e^{itK} O e^{-itK}$. For an operator of the form

$$O(t) := \int_A dx f(x, t) J(x) \quad \text{with } \text{supp}(f) \subset A, \quad (6.268)$$

we have the linear flow equation

$$\partial_t f(x, t) = -\beta(x) \partial_x f(x, t) - 2 \int_A dy \mathcal{N}(x, y) \partial_y f(y, t). \quad (6.269)$$

Then, if we start with $f(x, 0)$ localized in an open small interval inside the first interval A_1 and separated away from the boundary of A by a finite distance, the function $f(x, t)$ will spread everywhere in both intervals $A_1 \cup A_2$ for any $t \neq 0$. The expectation value $\langle O^\dagger(t)O(t) \rangle$ should be finite. From the correlator (6.96) it follows that the Fourier transform $\hat{f}(p, t) := \int_{-\infty}^{+\infty} f(x, t) e^{ipx} dx$ should satisfy $\int_{-\infty}^{+\infty} dp |\hat{f}(p, \tau)|^2 |p| < \infty$. This does not allow for a sharp discontinuity in the test function $f(x, t)$ at the boundary of A , since that would give $\hat{f}(p, t) \sim p^{-1}$. However, a term falling like the boundary behavior of $\mathcal{N}(x, y)$ (i.e. falling to zero with x as $x \rightarrow 0$ faster than any power of $\log(x)^{-1}$) can keep the test function in the space of allowed functions.

Of course, the eigenvectors of the modular Hamiltonian kernel diagonalize the modular flow. If we decompose the test function as

$$f(x, t) := \sum_{k=1}^2 \int_{-\infty}^{+\infty} ds \tilde{f}_k(s, t) u_s^k(x), \quad (6.270)$$

$$\tilde{f}_k(s, t) := \int_A dx v_s^k(x)^* f(x, t), \quad (6.271)$$

then, the flow equation gets diagonalized according to (6.234)

$$\tilde{f}_k(s, t) = e^{i2\pi st} \tilde{f}(s, 0). \quad (6.272)$$

6.3.2.6. Failure of duality for two intervals

For simplicity, in this section, we work in the unit circle S^1 picture. The intervals along the null-line are mapped into the unit circle using the inverse of the Cayley transformation (6.2). Let us consider two disjoint intervals $A_1, A_2 \subset S^1$. The complement of $A_1 \cup A_2$ is formed by two disjoint intervals, that we call A_3, A_4 . Then, we have that $(A_1 \cup A_2)' = A_3 \cup A_4$. If the global state is pure one usually assumes

$$S(A_1 \cup A_2) = S(A_3 \cup A_4), \quad (6.273)$$

which is equivalent to

$$I(A_1, A_2) = I(A_3, A_4) + (S(A_1) + S(A_2) - S(A_3) - S(A_4)). \quad (6.274)$$

Taking into account that the MI is a function of the cross ratio and that the entropies for single intervals are equal to the one of complementary intervals, we can express this relation as

$$I(\eta) = I(1 - \eta) + \frac{1}{6} \log \left(\frac{\eta}{1 - \eta} \right), \quad (6.275)$$

or, equivalently [26]

$$U(\eta) = U(1 - \eta). \quad (6.276)$$

That is, the symmetry property for the entropy of complementary regions (6.273) gives the symmetry of the function $U(\eta)$. This symmetry has also been shown as a consequence of modular invariance for two dimensional CFTs. We have seen this symmetry is not present for the free chiral current. This is not a problem of the continuum limit, the same happens in a finite lattice as we will show in the next section. As we have explained in section 4.2.2, this comes from the failure of the duality relation

$$\mathcal{A}(A_1 \cup A_2) = \mathcal{A}((A_3 \cup A_4)') \subsetneq \mathcal{A}(A_3 \cup A_4)'. \quad (6.277)$$

In fact, we can define the operator

$$\Phi_{12}^\lambda := e^{i\lambda \int_{\mathbb{R}} dx f(x) J(x)}, \quad (6.278)$$

where $\lambda \in \mathbb{R}$ and $f(x)$ is any smooth function such that

$$f(x) := \begin{cases} 1 & x \in A_3, \\ 0 & x \in A_4. \end{cases} \quad (6.279)$$

This operator does not belong to the algebra $\mathcal{A}(A_1 \cup A_2)$ but it belongs to $\mathcal{A}(A_3 \cup A_4)'$, since it commutes with all the operators belonging $\mathcal{A}(A_3 \cup A_4)$ because of the commutation relations (6.95). It is evident that any two operators like (6.278), with the same parameter $\lambda \in \mathbb{R}$ but given by two different smear functions $f(x)$ satisfying (6.279), will differ by an element of $\mathcal{A}(A_1 \cup A_2)$. Therefore, the choice of $f(x)$ satisfying these properties will not change the algebra $\mathcal{A}(A_1 \cup A_2) \vee \{\Phi_{12}^\lambda\}$. In fact, we have that

$$\mathcal{A}((A_1 \cup A_2)') = \mathcal{A}(A_1 \cup A_2) \vee \{\Phi_{12}^\lambda : \lambda \in \mathbb{R}\}. \quad (6.280)$$

We call the operator Φ_{12}^λ a *long-link* joining A_1 with A_2 , since being the exponential of the integral of $\partial_+ \phi(x)$, it is equivalent to some difference of the field ϕ localized inside the two intervals. However, in this model the field $\phi(x)$ does not actually exist, and the only way to write this operator is through an integral of the current as in (6.278).

There is still the possibility of considering a long link crossing the interval A_4 , rather than A_3 as in Φ_{12}^λ . However, a sum of these two long links is equivalent (modulo operators in $\mathcal{A}(A_1 \cup A_2)$) to the integral of the current along the whole unit circle $\oint dx J(x)$. This last operator commutes with all the algebra and it is multiple of the identity operator in the vacuum representation. Hence, the two options for the long link are actually equivalent.

Now, in an analogously way, we can define a long link crossing A_1 . For example, we

can define the operator

$$\Phi_{34}^{\lambda'} := e^{i\lambda' \int_{\mathbb{R}} dx g(x) J(x)}, \quad (6.281)$$

with $\lambda' \in \mathbb{R}$ and $g(x)$ is any smooth function such that

$$g(x) := \begin{cases} 1 & x \in A_1, \\ 0 & x \in A_2. \end{cases} \quad (6.282)$$

It follows from the commutation relations that

$$\Phi_{12}^{\lambda} \Phi_{34}^{\lambda'} = e^{i\lambda\lambda'} \Phi_{34}^{\lambda'} \Phi_{12}^{\lambda}. \quad (6.283)$$

Therefore, $\mathcal{A}((A_1 \cup A_2)')$ and $\mathcal{A}((A_3 \cup A_4)')$ do not commute. Hence, the theory does not satisfy essential duality (see definition 2.76), and the local algebras cannot be enlarged (with out spoiling locality) in order to obtain a theory satisfying the duality relation. Instead, we have that²⁶

$$\mathcal{A}(A_1 \cup A_2)' = \mathcal{A}(A_3 \cup A_4) \vee \left\{ \Phi_{34}^{\lambda'} : \lambda' \in \mathbb{R} \right\}, \quad (6.284)$$

$$\mathcal{A}(A_3 \cup A_4)' = \mathcal{A}(A_1 \cup A_2) \vee \left\{ \Phi_{12}^{\lambda} : \lambda \in \mathbb{R} \right\}. \quad (6.285)$$

In order to have duality for a two-interval region in this model, one should choose to add the long link to the algebra of one of the pairs of intervals (e.g. $A_1 \cup A_2$) and not for the complementary one (e.g. $A_3 \cup A_4$). This prescription necessarily does not treat in an equivalent way all pairs of intervals. Another perhaps more disturbing consequence of this choice is that the algebra $\mathcal{A}((A_1 \cup A_2)')$ containing the long link is not additive, meaning that it is not the generated algebra by the algebras of the two intervals. This is because the long link does not belong to the algebra generated by the single intervals, i.e.

$$\mathcal{A}(A_3 \cup A_4)' = \mathcal{A}((A_1 \cup A_2)')' \not\supseteq \mathcal{A}(A_1) \vee \mathcal{A}(A_2). \quad (6.286)$$

Hence, we can have additivity at the expense of duality or viceversa, but not both properties together. The natural choice is (6.100) because it can be consistently assigned to any pair of interval regions, and because of the additivity property, it is the only choice that allows the definition of the MI.

It is worth to notice that the free chiral fermion, which is an “extension” of the current algebra, satisfies (twisted) duality for two intervals and also additivity. This is why for the fermion $U(\eta) = U(1 - \eta)$ since trivially $U(\eta) = 0$. It is by reducing the theory to the current algebra that we run into this particular trouble.

This failure of duality for two intervals in general chiral conformal models has been

²⁶This can be easily shown in the finite lattice model of the next subsection.

associated with an algebraic μ -index of the inclusion of subalgebras $\mathcal{A}(A_3 \cup A_3) \subsetneq \mathcal{A}(A_1 \cup A_2)'$ [137]. This index “quantifies” the presence of superselection sectors in the theory, and should also determine the amount of asymmetry in the MI [62] as²⁷

$$U(0) - U(1) = \frac{1}{2} \log(\mu). \quad (6.287)$$

In the present model we have seen this is divergent, in accordance with the fact that the μ -index of the current algebra is infinity. In the following chapter we study this issue in general, finding a clear connection between EE and the superselection structure of the theory.

Restoration of the duality

From the point of view of a CFT in $d = 2$, it might seem strange that we could have duality violation for two intervals and hence $U(\eta) = U(1 - \eta)$ fails. This property can be derived from modular invariance of the twist operators giving place to the Rényi entropies [126]. The reason is that duality can be restored by adequately combining two chiral theories.

Let us look at the example of the massless limit of a free massive scalar. The usual local algebra for a massive scalar in $d = 2$ satisfies duality and additivity for two double cones, corresponding to two intervals in the spatial line at $t = 0$ (see chapter 5). In the massless case, the zero mode of the scalar field has divergent mean quadratic variation and has to be removed. However, the spatial and time derivatives of the field remain. With them we can form the two chiral currents $\partial_{\pm}\phi$. For a single diamond, the algebra is then equivalent to the one of two decoupled chiral currents in single interval regions. For two diamonds however, the algebra also contains the difference $\phi(x_2) - \phi(x_1)$, with x_1 and x_2 belonging to two different diamonds. We can take x_1, x_2 on the two intervals at $t = 0$, i.e.

$$\phi(x_2) - \phi(x_1) = \int_{x_1}^{x_2} dx \partial_x \phi(0, x) = \int_{x_1^+}^{x_2^+} dx^+ \partial_+ \phi(x) - \int_{x_1^-}^{x_2^-} dx^- \partial_- \phi(x). \quad (6.288)$$

Hence, the two diamonds contain the difference of long link operators corresponding to the two chiral algebras. However, they do not contain the sum of these long link operators, and therefore the chiralities do not decouple for two diamonds. This is the reason why these algebras for the two diamonds are compatible with the ones of the two complementary diamonds: the chiral long link operators do not commute to each other but their sum does, since commutators come out with opposite sign. Thinking in terms of the field differences (6.288), this commutation is evident. Therefore, these algebras have the same form for all pairs of diamonds and duality is restored. However, without the zero mode,

²⁷The paper [62] proves this relation for subalgebras of the free chiral fermion field, and conjectures its greater validity for chiral CFT models.

additivity is lost in this example. The massless limit of the MI of two intervals is divergent as $I \sim \frac{1}{2} \log(-\log(m))$ [105].

6.3.3. The free chiral current in the lattice

For the numerical calculation we put the model in a lattice. We take the lattice Hamiltonian

$$H := \frac{1}{2} \sum_{j \in \mathbb{Z}} f_j^2, \quad (6.289)$$

and the commutator

$$[f_j, f_l] = i(\delta_{l,j+1} - \delta_{l,j-1}) =: i C_{jl}. \quad (6.290)$$

Let us take a periodic system and, in order for C to be invertible, we take an even number $N := 2n$ of points. The eigenvectors of the commutator are given by phase factors

$$\sum_{l=1}^N C_{jl} e^{ikl} = 2i \sin(k) e^{ikj}, \quad k := \frac{2\pi m}{N}, \quad m = -(n-1), \dots, n. \quad (6.291)$$

From here, it follows that defining the following variables

$$\phi_k := \frac{1}{\sqrt{N \sin(k)}} \sum_{j=1}^N \cos(kj) f_j, \quad (6.292)$$

$$\pi_k := \frac{1}{\sqrt{N \sin(k)}} \sum_{j=1}^N \sin(kj) f_j, \quad (6.293)$$

where $k = \frac{2\pi m}{N} \in (0, \pi)$ with $m \in (1, n-1)$, we have that $[\phi_k, \pi_{k'}] = i \delta_{k,k'}$. We have also two other variables ψ_0, ψ_π that form the center of the global algebra since they commute with all other elements,

$$\psi_0 := \frac{1}{\sqrt{N}} \sum_{j=1}^N f_j \quad \text{and} \quad \psi_\pi := \frac{1}{\sqrt{N}} \sum_{j=1}^N (-1)^j f_j. \quad (6.294)$$

The inverse relation of (6.292-6.293) is

$$f_j = \frac{2}{\sqrt{N}} \sum_{m=1}^{n-1} \sqrt{\sin(k)} (\cos(kj) \phi_k + \sin(kj) \pi_k) + \frac{\psi_0}{\sqrt{N}} + \frac{\psi_\pi (-1)^j}{\sqrt{N}}, \quad (6.295)$$

where $k = \frac{2\pi m}{N} \in (0, \pi)$. The Hamiltonian writes in these new variables as

$$H = \sum_{m=1}^{n-1} \sin(k) (\phi_k^2 + \pi_k^2) + \frac{1}{2} \psi_0^2 + \frac{1}{2} \psi_\pi^2. \quad (6.296)$$

This gives for the vacuum state $\langle \phi_k^2 \rangle = \langle \pi_k^2 \rangle = \frac{1}{2}$ and $\langle \phi_k \pi_k \rangle = i/2$. The center can take any value and we set $\psi_0 = \psi_\pi = 0$. Hence, we impose these relations as a constraint. In this way, we get a pure vacuum state and a global algebra without center. The full system has now $n - 1$ degrees of freedom: $n - 1$ coordinates and $n - 1$ momentum variables.

We see from (6.296) that we have two sets of low energy degrees of freedom, for $k \sim 0$ and $k \sim \pi$. Thereby, the system shows a doubling of the degrees of freedom in the continuum, analogous to the usual fermion doubling. This is the reason we also have two commuting operators ψ_0 and ψ_π .

The correlator of the original variables $F(j - l) = \langle f_j f_l \rangle$ is given by

$$F(x) = \begin{cases} \frac{1}{N} \frac{\cos^2\left(\frac{\pi x}{2}\right) \sin\left(\frac{2\pi}{N}\right)}{\sin\left(\frac{\pi(x+1)}{N}\right) \cos\left(\frac{\pi(x-1+N/2)}{N}\right)}, & |x| \neq 1 \\ \frac{i}{2} C(x), & |x| = 1 \end{cases}$$

In the limit of a large circle $N \rightarrow \infty$, we have that

$$F(x) = \begin{cases} -\frac{1+(-1)^x}{\pi(x^2-1)}, & |x| \neq 1 \\ \frac{i}{2} C(x), & |x| = 1 \end{cases} \quad (6.297)$$

The entropy of a region follows from (4.84) and (4.87). We first check numerically the entropy for a single interval. We calculate the matrices (6.290) and (6.297) for intervals of length $R = 10k$ with $k = 1, \dots, 20$. We fit the pairs $(R_k, S(R_k))$ with $c_0 + c_{\log} \log k + c_{-1} \frac{1}{k} + c_{-2} \frac{1}{k^2}$ obtaining the logarithmic coefficient $c_{\log} = 1/3$ with high precision. Notice that this coefficient is twice the expected one for the chiral current model. This reflects the doubling on the lattice.

To calculate the MI between two intervals of length a and b separated by a distance c , that is $I(A, B) = S(A) + S(B) - S(A \cup B)$, we need the entropies of the single intervals $S(A)$ and $S(B)$, and the entropy of the two intervals $S(A \cup B)$. Each of these entropies is calculated using (4.84) and (4.87). In the continuum limit the MI is a function of the cross-ratio η , where η is defined as

$$\eta := \frac{a \cdot b}{(a + c)(b + c)}, \quad (6.298)$$

in accordance to (6.173). For a given cross-ratio, we repeat the calculation for different configurations that differ one from another just by a dilatation with parameter $k = 2, 4, \dots, 20$. We then fit the pairs $(k, I_k(\eta))$ with $c_0 + c_{-1} \frac{1}{k} + c_{-2} \frac{1}{k^2} + c_{-3} \frac{1}{k^3}$, and take the constant coefficient c_0 as the continuum limit of the mutual information for the lattice model, which is twice the chiral current model due to doubling. We then take $I(\eta) = c_0/2$. We repeat the same procedure for different values of η obtaining the red points showed in figure 6.2.

In doing simulations for this model it is important that, if N is finite, we take the total number of points even $N = 2n$, and, in order to not have a center, the subsystems need to have an even number of points or variables, i.e, intervals of even size. This is because half of them are coordinates and half are momenta. The complementary subsystems automatically must have equal entropy because the global state is pure. For example, for an interval of size $2k$ in a circle of size $2n$, the commutant is an interval of size $2n - 2k - 2$, because there are two points in the complementary region adjacent to the interval that do not commute with the original interval. The entropies are indeed equal. When we consider two-interval regions the commutant algebra contains a long link as explained in the previous section. In the lattice, it contains two long links operators because of the doubling. More precisely, these commutant algebras with long links for two intervals are of the form: all points in the intervals $(a_1, b_1) \cup (a_2, b_2)$, and two long links given by the sums $\sum_j f_j$ and $\sum_j (-1)^j f_j$, where the sums are over all the points in the open interval $(b_1, a_2) := (b_1 + 1, \dots, a_2 - 1)$. The long links crossing the other gap between the intervals are related to these by elements of the algebra of the intervals and the global constraints, and hence they do not give additional operators. The counting of degrees of freedom is as follows: for a circle of $2n$ points, if the original intervals have $2k_1$ and $2k_2$ points, the commutant will have $2n - 2k_1 - 2k_2 - 4$ points plus two long links. This gives a total of $2n - 2k_1 - 2k_2 - 2$ linearly independent operators. This is precisely (twice) the complementary number of degrees of freedom: $2n - 2$ is twice the total number of degrees of freedom in the lattice.

We have checked the entropies of complementary algebras of two intervals are equal in the circle. The entropy for the two intervals with the long links \tilde{S} can also be completed to form a “kind of mutual information”, eliminating UV divergences in the continuum, as

$$\tilde{I}(A_1, A_2) = S(A_1) + S(A_2) - \tilde{S}(A_1 \cup A_2). \quad (6.299)$$

The equality of the entropies for two intervals and the one of the complementary region including the long links

$$S(A_1 \cup A_2) = \tilde{S}(A_3 \cup A_4), \quad (6.300)$$

can be completed with single interval entropies to form a relation between the mutual informations

$$\tilde{I}(\eta) = I(1 - \eta) + \frac{1}{6} \log \left(\frac{\eta}{1 - \eta} \right). \quad (6.301)$$

We can define the U function for the entropies with the long link

$$\tilde{I}(\eta) = -\frac{1}{6} \log(1 - \eta) + \tilde{U}(\eta). \quad (6.302)$$

Then relation (6.301) is just the complementary relation for the $U(\eta)$

$$\tilde{U}(1 - \eta) = U(\eta). \quad (6.303)$$

These two should be symmetric and equal for a model satisfying duality for two intervals, but this is not the case of the present model. We have also checked numerically the relation (6.301) in the infinite lattice limite. For that, we calculate $\tilde{I}(\eta)$ and we note that the convergence to the continuum limit is much improved for this case using the fitting function as $c_0 + c_{-1/2} \frac{1}{k^{1/2}} + c_{-3/2} \frac{1}{k^{3/2}}$, instead of using integer powers, as long as we increase the global size k of the region. The continuum limit again corresponds to the coefficient $\tilde{I}(\eta) = \frac{c_0}{2}$.

6.4. Conclusions of the chapter

We obtained the vacuum modular Hamiltonian for the free chiral current in two intervals. The modular Hamiltonian contains the usual local term given by an integral of the energy density times a position dependent inverse temperature. This term is identical to the free chiral fermion one, and very probably it is universal for all theories in $d = 2$. Besides, there is a non-local term. This is given by a quadratic expression in the current with a locally integrable kernel which does not vanish in any open subset of $A \times A$. Hence, the modular Hamiltonian is completely non-local in contrast to the fermion case. The MI does not have the symmetry property (6.275), and the origin of this is the failure of duality for two intervals.

We treated the case of two intervals. More intervals could in principle be treated in a similar fashion, but the expressions will depend on a higher number of cross-ratios, and besides, the Hypergeometric and Appell functions that parametrize the eigenvectors should be replaced by higher dimensional Lauricella functions.

It would be interesting to understand why the free chiral fermion modular Hamiltonian is “quasilocal” while the one of the chiral current is completely non-local. The technical reason is that one of the eigenvectors of the bosonic model has a dependence on the eigenvalue s that is not simply a phase factor $e^{is\omega(x)}$. For the fermion field and any number of intervals, or the current field in a single interval, this same phase factor determines completely the dependence of all eigenvectors in s . Perhaps a reason for the fermion to be special is the multi-local symmetries described by Rehren and Tedesco [138].

We have shown that the current MI is smaller than the fermion one because the former model is a subalgebra of the later. It would be interesting to explore other consequences of this inclusion. For example, the difference of modular Hamiltonians $K_\psi - K_J$ between these models should be a positive operator. We can compute the expectation value of this

difference of operators in a state generated from the vacuum by acting with a unitary in A , for example, a coherent state $e^{i \int dx f(x) J(x)} |0\rangle$. The local contribution vanishes in the difference of expectation values of the two modular Hamiltonians, and we get an inequality involving exclusively the non-local parts of K_ψ and K_J .

Chapter 7

Entanglement entropy and superselection sectors

As we have explained in section 2.1.4, given an algebra of observables \mathfrak{A} , its space of pure states decomposes into superselection sectors (SS). These SS are related to unitarily inequivalent representations of \mathfrak{A} . In any given (non-irreducible) representation π of \mathfrak{A} , pure states belonging to the same superselection sector are related by elements of $\pi(\mathfrak{A})$. On the other hand, there is no operator in $\pi(\mathfrak{A})$ connecting pure states belonging to different SS.

In QFT, one is usually interested in states which are constructed from the vacuum one through local perturbations by elements of \mathfrak{A} . Hence, it seems plausible to restrict the attention to the vacuum GNS-representation $\mathcal{A} := \pi_0(\mathfrak{A}) \subset \mathcal{B}(\mathcal{H}_0)$, and dispense with the structure of the other states that the model \mathfrak{A} admits. However, it is the result of a large body of research into the superselection structure of QFT that the SS leave a definite imprint in the relations between the different local subalgebras assigned to regions in the model \mathcal{A} itself. More precisely, there is a subclass of sectors representing localizable states, such that they can be fully reconstructed from the vacuum sector [46, 48, 50, 72]. This superselection structure could be accomplished with the help of a bigger local QFT $\mathfrak{F} \subset \mathcal{B}(\mathcal{H})$, called the field algebra,¹ such that it contains copies of all the other representations corresponding to superselection sectors of localizable states. In this way, the Hilbert space \mathcal{H} decomposes as a direct sum of subspaces, where the observable algebra \mathcal{A} acts irreducibly. In such a decomposition, the vacuum representation $(\mathcal{A}, \mathcal{H}_0)$ appears once. The observable algebra \mathcal{A} could be obtained from \mathfrak{F} as the pointwise invariant part under the action of a compact symmetry group G . States belonging to the vacuum subspace $\mathcal{H}_0 \subset \mathcal{H}$ are invariant (neutral) with respect to the action of G , whereas vector states belonging to any other subspaces transform non-trivially under G , and hence they define *charged states*.

¹Traditionally, the algebra \mathcal{A} is thought to be the algebra of local physical observables, while the charged operators in \mathfrak{F} retain some locality properties but are not physically realizable in local laboratories, e. g. an operator that can change the baryonic number. In the theoretical setting of this thesis, we do not make this epistemological distinction.

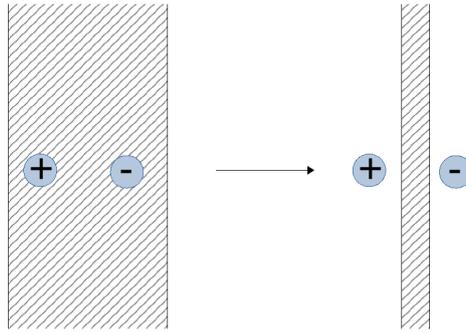


Figure 7.1: Mutual information between two regions separated by a strip of width $\epsilon > 0$ (shaded region in the figure). For ϵ wide enough, the typical charge-anticharge fluctuations are not sensed by the mutual informations of both models (left panel). When the width ϵ becomes small enough to allow for charge-anticharge fluctuations to occur on each side of the wall with enough probability (right panel), the MI of \mathfrak{F} will take into account these correlations while the one of the neutral model \mathcal{A} will not.

The elements of \mathcal{A} cannot transform between vectors in disjoint subspaces. However, there are charged local operators in \mathfrak{F} , which allows us to create any charged state from the vacuum subspace \mathcal{H}_0 .

This superselection structure affects the relations between algebras and regions in \mathcal{A} , either violating the property of duality and/or additivity for some topologically non-trivial regions (see definition 2.75). In this way, a theory having non-trivial SS is considered as an incomplete model. The superselection structure also affects the vacuum fluctuations through charge-anticharge virtual pairs. The EE should be sensitive to these behaviors. Therefore, the main focus of this chapter is the analysis of the consequences of the SS for the EE.

Localizable charged states remarkably come in two types, corresponding to global or gauge symmetry charges. These correspond to two abstract types of superselection sectors, called DHR (because of Haag, Doplicher, and Roberts [45–50]) and BF sectors (because of Buchholz and Fredenhagen [139]) respectively. The main difference between these two cases is geometric, global charges creating operators can be localized in a compact region, while gauge charges creating operators can be only localized in cones, allowing the Wilson line extends to infinity. Here we deal only with DHR SS.

An essential feature of the EE in QFT is that it cannot be defined without the introduction of an ultraviolet regulator, making this quantity inherently ambiguous through the regularization scheme choice. As we have explained in section 4.2, this ambiguity can be cured by computing half the MI between nearly complementary regions, which are separated by a regulating distance $\epsilon > 0$. This is a natural quantity, taking the place of EE, that is well-defined in the continuum model and it is used as a universal regularized EE. On the other hand, there is also another source of ambiguities that has been discussed in the literature concerning the assignation of local algebras to regions. In this sense, local

algebras may contain a center related to ambiguities in the choice of the algebra at the boundary of the region in a lattice model. This type of ambiguities has attracted especially attention in relation to gauge models (for the discussion around this topic see for example [140–148]). However, this kind of local ambiguities does not survive the continuum limit, leaving the MI as a well-defined quantity [149]. In the new scenario we are presenting here, where models with SS sectors are considered, the analysis is enriched giving place to more interesting consequences. In models with SS, there is more than one choice of the algebra for topologically non-trivial regions. The MI is sensitive to these possible choices. However, these mutual informations can be reinterpreted as corresponding to different models, with and without SS.

More concretely, the MI crucially depends on the physical regulating distance ϵ that allows us to sense or not the presence of virtual charge pairs according to the comparison of the size of ϵ with the typical scale Λ of these fluctuations (see figure 7.1). Hence, two possible results may come out in the limits $\epsilon/\Lambda \gg 1$ or $\epsilon/\Lambda \ll 1$, independently of the size R of the region, whenever R is much larger than both ϵ and Λ . Then, in terms of the MI, it may seem we still have an apparent ambiguity. One of the main results that come from the analysis of SS is the clarification of this issue. Each result for the mutual information corresponds to a particular choice of the algebra, where the SS have been included or not respectively. This means, we should not interpret this as an ambiguity but as a consequence of alternative model choices.

With this perspective, partially following previous works in the mathematical literature [62, 150], we develop entropic order parameters capable to sense these differences. In this way, we study models with finite or Lie group symmetry groups, and spontaneous symmetry breaking symmetries.

While throughout this chapter we try to keep the discussion as simple and physical as possible, with a mixed degree of mathematical rigor, we are forced to use some specific mathematical tools to avoid making ambiguous statements. We do not treat explicit examples where the superselection sectors do not come from a symmetry group. This includes models with DHR sectors in $d = 2$. This would require more formal developments but would not add any new to the general physical picture.

This chapter is structured as follows. In section 7.1, we describe the problems in the relations between algebras and regions in theories with superselection sectors, and we introduce the main elements of the theory of DHR superselection sectors. In section 7.2, we investigate the EE in the case of DHR sectors, describing the relevant entropic order parameters and their mutual relationships, that take the form of entropic certainty and uncertainty relations. We explicitly compute the relevant quantities in several cases of interest. This includes the cases of finite and Lie symmetry groups, compactified scalars, regions with different topologies and charge excitations. In section 7.3, we study in con-

crete examples, the behavior of the expectation values of some operators (intertwiners and twists), which are the main witnesses of the superselection sectors and play a central role in the evaluation of the entropic quantities. We end, in section 7.4, with a summary and the conclusions.

7.1. Algebras, regions and superselection sectors

We are interested in some particularities of the relation between algebras and regions that affect the EE. To start the discussion, we assume that our observable QFT is defined as net of vN algebras

$$\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0), \quad (7.1)$$

in the vacuum representation, satisfying all the axioms of definitions 2.56 and 2.58. We also assume that the vacuum vector $|0\rangle \in \mathcal{H}_0$ is the unique Poincaré invariant vector in the Hilbert space \mathcal{H}_0 . Moreover, we assume that such a vacuum representation is unique. Weak additivity (definition 2.61) is also assumed for the observable algebra \mathcal{A} .

Remark 7.1. In the following subsection, we slightly modify these assumptions in order to study the SS structure of the theory.

Among the above minimal assumptions, we want to remark isotony

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2), \quad \text{if } \mathcal{O}_1 \subset \mathcal{O}_2, \quad (7.2)$$

and causality,

$$\mathcal{A}(\mathcal{O}') \subset \mathcal{A}(\mathcal{O})'. \quad (7.3)$$

We could also ask whether the observable algebra would satisfy the stronger assumptions developed in section 2.2.5. More concretely, we are interested in considering the assumptions of the weaker version of definition 2.75, where 2 is replaced by 2', or equivalently the assumptions of definition 2.84 once a Cauchy surface is chosen. We remark here again that these assumptions are expected to hold only for “sufficiently complete” theories. However, as we will see below, some of them fail for physically significant models.

Duality condition (4 in definition 2.75) is an enhancement of the causality property above. When the duality condition does not hold, it may seem rather simple to complete a given net of algebras to have duality, just by enlarging the algebras taking $\mathcal{A}(\mathcal{O}) \rightarrow \mathcal{A}^d(\mathcal{O}) = \mathcal{A}(\mathcal{O}')'$. However, in this process of completion, some problems might arise, since we are enlarging the algebra of the region \mathcal{O} and its complement \mathcal{O}' at the same time. In particular, it may happen that $\mathcal{A}^d(\mathcal{O})$ and $\mathcal{A}^d(\mathcal{O}')$ do not commute. Moreover, the enlarged algebra may not satisfy additivity and/or intersection property (2' and 3 in

definition 2.75) even though the original algebra does.

Additivity property means that algebra of a region \mathcal{O} could be generated by the algebras of operators of smaller regions included in \mathcal{O} . This expresses that the algebra is locally generated. In the Garding-Wightman approach based on point-like localized fields, this is how one would form the algebra of a region \mathcal{O} , by taking arbitrary polynomials of smeared fields with support in arbitrary small regions inside \mathcal{O} . Intersection property is a sort of dual version of additivity. According to the discussion in section 2.2.5, if the theory satisfies duality for any causally complete region, then, due to the De Morgan laws, additivity holds for \mathcal{O}_1 and \mathcal{O}_2 if and only if intersection property holds.

It is important to emphasize that when the duality condition fails, it could be the case that it still holds for a restricted subset of causally complete regions. For example, we will assume above that it holds for topological trivial regions, such as double cones. However, in a specific model, the imposition of (7.2) and (7.3) prevents duality to hold for non-trivial topological regions. This means that the failure of the duality condition must be considered as a physical fact and not as a sort of mathematical unphysical issue due to an inadequate election of algebras. Moreover, in this case, the different algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}^d(\mathcal{O})$ give place to different macroscopic statistical behaviors even for the vacuum state, which are detectable with the use of entanglement measures.

Summarizing, properties (7.2) and (7.3) are elementary axioms which always hold for any QFT model (see definition 2.56). However, duality, additivity and intersection property are assumptions that will hold only for “sufficiently complete” models, and fail for some physically significant models. Here we are interested in such failures due to the presence of non-trivial SS. We will see a simple example below.

It is important to realize that the failure of these properties does not (necessarily) have to do with the fact that QFT is a continuum theory with infinitely many degrees of freedom or the intricate nature of the specific type III vN algebras. The problems we want to address here are not related to UV physics, but they already appear in lattice models.

Notation 7.2. Throughout this chapter, we usually consider causally complete regions which are the Cauchy development of spacelike regions over the Cauchy surface of fixed time $x^0 = 0$. We denote such a Cauchy surface by Σ_0 . Moreover, we frequently use causally complete regions $D(\mathcal{C}_R)$, where \mathcal{C}_R is the sphere of radius R

$$\mathcal{C}_R := \{(0, \vec{x}) \in \Sigma_0 : |\vec{x}| < R\} . \quad (7.4)$$

Notation 7.3. To lighten the notation, the vacuum state is denoted by ω instead of ω_0 , as we have used before.

Example: bosonic subnet of the free fermion field

In order to introduce the main ideas, we find useful to start with a specific simple model, where all the relevant objects can be displayed explicitly. Once the main ideas are understood, we will discuss the general case. Let us then focus on the following model. We take a free Dirac field in d dimensions. This theory is defined by means of the local fields

$$\psi_j(x), \psi_k^\dagger(x), \quad j, k = 1, \dots, n := 2\lfloor \frac{d}{2} \rfloor, \quad (7.5)$$

which satisfy the canonical anticommutation relations²

$$\{\psi_j(x), \psi_k(y)\} = \{\psi_j^\dagger(x), \psi_k^\dagger(y)\} = 0, \quad (7.8)$$

$$\{\psi_j(x), \psi_k^\dagger(y)\} = iS_{jl}(x-y)\gamma_{lk}^0, \quad S(x) = (i\cancel{\partial} + m)\Delta(x), \quad (7.9)$$

where $\cancel{\partial} := \gamma^\mu \partial_\mu$, $\gamma^\mu \in \mathbb{C}^{n \times n}$ ($\mu = 0, \dots, d-1$) are the Gamma matrices (in some representation), and $\Delta(x)$ is the commutator function of the scalar field (see equation (5.86)). For any test function $f := (f_1, \dots, f_n)^T$ with $f_j \in \mathcal{S}(\mathbb{R}^d)$, the smeared operators are denoted by

$$\psi(f) := \int_{\mathbb{R}^d} d^d x f(x)^\dagger \psi(x) \quad \text{and} \quad \psi^\dagger(f) := \int_{\mathbb{R}^d} d^d x \psi^\dagger(y) f(x), \quad (7.10)$$

which are bounded operators acting on the fermionic Fock Hilbert space \mathcal{H} . The local algebras for a causally complete region $\mathcal{O} \in \mathcal{K}$ are defined as the vN algebras

$$\mathfrak{F}(\mathcal{O}) := \{\psi(f_1) + \psi^\dagger(f_2) : \text{supp}(f_j) \subset \mathcal{O}\}'' . \quad (7.11)$$

The fermionic net satisfies the axioms 1, 2', and 3-5 of definition 2.75, including additivity and (twisted) duality [151].³

However, we are interested in the bosonic subnet of the fermion net, i.e. the subalgebra $\mathcal{A} \subset \mathfrak{F}$ consisting of all operators with even fermionic number. It is generated by all operators made out of an even number of fermion fields, i.e. $\mathbf{1}$, $\psi_j(x)\psi_k(y)$, $\psi_j^\dagger(x)\psi_k^\dagger(y)$, $\psi_j(x)\psi_k^\dagger(y)$, etc. All these fields have to be smeared with test functions, and we can take arbitrary polynomials with even fermionic number. This subalgebra acts naturally in the subspace $\mathcal{H}_0 \subsetneq \mathcal{H}$ formed by the states with even number of

²In the free theory, the time-zero fields $\psi_j(0, \bar{x})$ and $\psi_j^\dagger(0, \bar{x})$ are well-defined operators valued distributions in the fermionic Fock Hilbert space (see section 2.2.3 and [41]). In this case, (7.8-7.9) imply

$$\{\psi_j(0, \bar{x}), \psi_k(0, \bar{y})\} = \{\psi_j^\dagger(0, \bar{x}), \psi_k^\dagger(0, \bar{y})\} = 0, \quad (7.6)$$

$$\{\psi_j(0, \bar{x}), \psi_k^\dagger(0, \bar{y})\} = \delta(\bar{x} - \bar{y}). \quad (7.7)$$

³For fermionic nets, we must consider *twisted duality* $\mathfrak{F}(\mathcal{O}') = \mathfrak{F}(\mathcal{O})^{t'}$ (see section 2.2.4), instead of assumption 4 of definition 2.75.

(fermionic) particles. Moreover, the vacuum vector $|0\rangle \in \mathcal{H}_0$ is cyclic for the algebra \mathcal{A} in the bosonic Hilbert space \mathcal{H}_0 . Regardless we have defined the bosonic net as a subnet of the fermion net, we want to emphasize that the former exists in its own right. In fact, it could be defined without referring to a bigger algebra nor a bigger Hilbert space. The bosonic net \mathcal{A} is called the algebra of observables, since it is formed uniquely by "observables quantities", which commute at spacelike distance. Remarkably, all the physics is encoded in the observable algebra itself from which the bigger algebra \mathfrak{F} could be reconstructed. We will explain this fact more detailed in section 7.1.2.

Once we have recognized the global algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_0)$, we are interested in the assignation of subalgebras to regions $\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0)$. For a double cone $W \subset \mathbb{R}^d$, we define

$$\mathcal{A}(W) := \{ [\psi(f_1) + \psi^\dagger(f_2)] [\psi(f_3) + \psi^\dagger(f_4)] : \text{supp}(f_j) \subset W \}'' , \quad (7.12)$$

where the double commutant is taken inside $\mathcal{B}(\mathcal{H}_0)$. At this point, it could be intriguing why we have decided to use definition (7.12) only for double cones, and not for any causally complete region. The reason is that for more general regions, there is more than one alternative. For example, let us take two spacelike separated double cones W_1 and W_2 . If we define the algebra $\mathcal{A}(W_1 \vee W_2)$ using (7.12), we have that the theory would not satisfy additivity. In fact, the "intertwiner" operator

$$\mathcal{I}_{12} := \psi(f_1) \psi(f_2) , \quad \text{supp}(f_j) \subset W_j , \quad (7.13)$$

belongs to the global algebra \mathcal{A} and would belong to the algebra of the $W_1 \vee W_2$ according to (7.12). However, it does not belong to the additive algebra $\mathcal{A}(W_1) \vee \mathcal{A}(W_2)$. This is because of the presence of a dual operator of \mathcal{I}_{12} , called the "twist" operator. In order to give a simple expression for such an operator, let us assume for a moment that $W_1 := D(\mathcal{C}_R)$. Then, the "twist" operator is defined as^{4,5}

$$\tau := e^{i\pi \int d^d x g(x^0) h(\bar{x}) : \psi^\dagger(x) \psi(x) :} , \quad (7.15)$$

⁴The reason why these operators are called "intertwiner" and "twist" will be clearer in the next section.

⁵A general double cone can be represented as $W_1 := D(\mathcal{C})$, where $\mathcal{C} \subset \Sigma$ is a spacelike region over some Cauchy surface Σ . A twist operator for W_1 can be defined as

$$\tau := e^{i\pi \int d^d x g(x) : \bar{\psi}(x) \gamma^\mu \psi(x) : n_\mu} , \quad (7.14)$$

where n_μ is the unit normal future pointing timelike vector to Σ and $g(x)$ is an appropriate smearing function.

where⁶

$$g \in C_c^\infty(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} g(t) dt = 1, \quad (7.16)$$

$$h \in C_c^\infty(\mathbb{R}^{d-1}, \mathbb{R}) \quad \text{and} \quad h(\bar{x}) = 1 \quad \forall \bar{x} \in \mathcal{C}_R, \quad (7.17)$$

$$\text{supp}(gh) \not\ll W_2. \quad (7.18)$$

The twist is a unitary operator, which belongs to the global algebra \mathcal{A} , trivially commutes with $\mathcal{A}(W_2)$, and anticommutes with the fermion operators in W_1 , i.e. $\tau\psi(f)\tau^{-1} = -\psi(f)$ for all smearing functions with $\text{supp}(f) \subset W_1$. Then, τ also commutes with $\mathcal{A}(W_1)$. In other words, $\tau \in (\mathcal{A}(W_1) \cup \mathcal{A}(W_2))'$. However, τ does not commute with \mathcal{I}_{12} . In fact, we have that

$$\{\tau, \mathcal{I}_{12}\} = 0 \Rightarrow [\tau, \mathcal{I}_{12}] \neq 0. \quad (7.19)$$

Because of this, we have that \mathcal{I}_{12} cannot belong to the additive algebra $\mathcal{A}(W_1) \vee \mathcal{A}(W_2) = (\mathcal{A}(W_1) \cup \mathcal{A}(W_2))''$.

On the other hand and in order to preserve additivity, we can define the algebra of $W_1 \vee W_2$ as

$$\mathcal{A}(W_1 \vee W_2) := \mathcal{A}(W_1) \vee \mathcal{A}(W_2). \quad (7.20)$$

We will show now that such a choice does not satisfies duality. In fact, if we want to preserve additivity for any causally complete region, the algebra of the of complement of the union of two double cones $(W_1 \vee W_2)'$, has to be defined according (7.12), i.e.

$$\mathcal{A}((W_1 \vee W_2)') = \bigvee_{W \subset (W_1 \vee W_2)'} \mathcal{A}(W), \quad (7.21)$$

where the union runs along all the double cones W included in $(W_1 \vee W_2)'$. It is important to emphasize that the above definition is the only possible one if we want to preserve additivity. However, this algebra does not contain the operator (7.15), because it cannot be formed additively inside $(W_1 \vee W_2)'$. Or more precisely, the operator \mathcal{I}_{12} commutes with every local algebra $\mathcal{A}(W)$ with $W \subset (W_1 \vee W_2)'$, and hence

$$\mathcal{I}_{12} \in \left(\bigcup_{W \subset (W_1 \vee W_2)'} \mathcal{A}(W) \right)'. \quad (7.22)$$

⁶In other words, the function $g(x^0)h(\bar{x})$ must be any smooth regularization of the function $\delta(x^0)\Theta_{\mathcal{C}}(\bar{x})$ with support strictly spacelike separated to W_2 .

Then, we have that $\tau \notin \mathcal{A}((W_1 \vee W_2)')$ because of (7.19) and (7.22). This implies that

$$\mathcal{A}((W_1 \vee W_2)') \subsetneq \mathcal{A}(W_1 \vee W_2)' = \mathcal{A}((W_1 \vee W_2)') \vee \{\tau\}, \quad (7.23)$$

showing explicitly the failure of the duality condition for the union of two double cones. On the other hand, for any connected region $\mathcal{O} \in \mathcal{K}$, the additively generated algebra $\bigvee_{W \subset \mathcal{O}} \mathcal{A}(W)$ coincides with the one defined through 7.12. Moreover, for such a connected region duality holds. In other words, in this model, the ambiguity between algebras and regions, and the failure of the duality condition are attainable only to non-connected regions.

The intersection property also conflicts with additivity since we can take two connected causally complete regions $\mathcal{O}_1, \mathcal{O}_2$, but whose intersection is the union of two disconnected regions \mathcal{O}_3 and \mathcal{O}_4 , i.e. $\mathcal{O}_1 \cap \mathcal{O}_2 = \mathcal{O}_3 \cup \mathcal{O}_4$. It not difficult to see, that an intertwiner \mathcal{I}_{34} (as the one of (7.13)) between \mathcal{O}_3 and \mathcal{O}_4 belongs to the additive algebras $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$, and therefore, it belongs to the intersection of the algebras $\mathcal{A}(\mathcal{O}_1) \wedge \mathcal{A}(\mathcal{O}_2)$. However, as we have discussed above, \mathcal{I}_{34} does not belong to the additive algebra $\mathcal{A}(\mathcal{O}_3) \vee \mathcal{A}(\mathcal{O}_4)$. In other words, if we accept the intersection property, the intertwiner will belong to the algebra of $\mathcal{O}_3 \cup \mathcal{O}_4$, and this algebra will not be additive.

To summarize, if we start with the operators belonging to $\mathcal{A}(W_1) \cup \mathcal{A}(W_2)$, its commutant will contain the twist operator, and then, the double commutant will not contain the intertwiner. Conversely, if we start with the algebra of two double cones plus the intertwiner $\mathcal{A}(W_1) \cup \mathcal{A}(W_2) \cup \{\mathcal{I}_{12}\}$, its commutant will not contain the twist. In other words, denoting by $\mathcal{A}_{\mathcal{O}} := \mathcal{A}(\mathcal{O})$ the additive algebra for any region $\mathcal{O} \in \mathcal{K}$, then we have

$$\mathcal{A}'_{W_1 \vee W_2} = \mathcal{A}_{(W_1 \vee W_2)'} \vee \{\tau\}, \quad (7.24)$$

$$\mathcal{A}'_{(W_1 \vee W_2)'} = \mathcal{A}_{W_1 \vee W_2} \vee \{\mathcal{I}_{12}\}. \quad (7.25)$$

Therefore, duality for the union of two double cones (resp. the complement of two double cones) would require that we enlarge the additive algebra of its complementary region with the twist operator (resp. intertwiner), losing additivity.⁷ Also, this possible enlargement does not treat in the same way the algebra of a region and the one of its commutant. Notice that we could change the definition of the intertwiner and/or the twist by choosing different smearing functions satisfying the stated requisites. But, these different operators differ between them by elements of the additive algebra.

From now on, $\mathcal{A}_{\mathcal{O}}$ will always denote the additive algebra of the region $\mathcal{O} \in \mathcal{K}$, which

⁷This is exactly what happen for the free chiral current, as we have discussed in the previous chapter (see section 6.3.2.6).

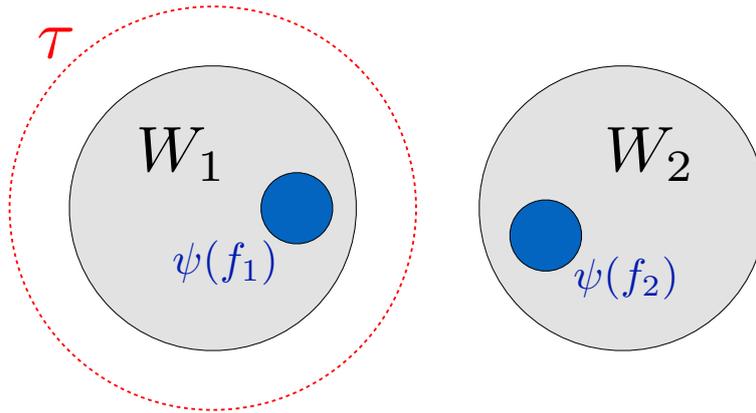


Figure 7.2: The intertwiner $\mathcal{I}_{12} = \psi(f_1)\psi(f_2)$ commutes with the additive algebra of the exterior of the two balls W_1 and W_2 . It cannot be formed additively with the algebras of these balls. The twist τ cannot be formed additively on the exterior of the two balls but commutes with the algebras of the balls. The intertwiner and the twist do not commute with each other.

by definition is

$$\mathcal{A}(\mathcal{O}) = \bigvee_{\alpha} \mathcal{A}(W_{\alpha}), \quad (7.26)$$

where the union runs over any family of double cones such that $\mathcal{O} = \bigcup_{\alpha} W_{\alpha}$. In a general theory, the above relation will be always assumed, i.e. we always assume that our net of observables satisfies additivity.

Relation (7.25), which “measures” the violation of the duality relation for two double cones, plays a central role in this chapter since it shows explicitly the possible ambiguities in the assignation of algebras to regions. In fact, the difference between the two choices of algebras is due to the existence of the intertwiner operator \mathcal{I}_{12} . The following sections aim to show that the violation of the duality for two double cones occurs in any QFT having non-trivial superselection sectors. This relation between the failure of the duality condition and the existence of SS will allow us to give a general and concrete description of the difference of the algebras $\mathcal{A}'_{(W_1 \vee W_2)'} and $\mathcal{A}_{W_1 \vee W_2}$.$

7.1.1. DHR superselection sectors

Here we start with an AQFT given by a net of C^* -algebras $\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{A}$. As we have discussed in section 2.1.4, the space of pure states $\mathfrak{S}_p(\mathcal{A})$ of the global algebra \mathcal{A} decomposes into coherent subsets called sectors. Between the collection of all sectors, there is a particular subcollection, called DHR sectors, because of the work of Doplicher, Haag and Roberts [45–50] (for a review see [2, 57, 58]). These sectors, whose definition is given by criterion 7.4 below, contain most of the states which are interesting in elementary particle physics. These states are such that they behave asymptotically like the vacuum state for observations in far away regions of the space. It turns out that the set of all DHR pure states is larger than the collection of vector states in one irreducible representation

(e.g. the vacuum representation), but certainly, smaller than the set of all pure states over \mathcal{A} . For our interest, we will show that the existence of non-trivial DHR sectors is the cause of the failure of the duality for two double cones.

Recall the connection between states and Hilbert space representations we explained in section 2.1.4. Each representation has an affiliated family of normal states, which are given by the density matrices on the representation Hilbert space. The pure states affiliated with one irreducible representation form a superselection sector. Pure states affiliated with inequivalent irreducible representations belongs to different sectors. Then, instead of describing “the states of interest” we may thus equally well describe the “classes of representations of interest”.

Definition 7.4. Let $\pi_0 : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_0)$ be the vacuum representation of \mathcal{A} . A *DHR representation* is any representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ which is unitarily equivalent to the vacuum one outside any double cone, or in other words

$$\pi|_{\mathcal{A}(W')} \cong \pi_0|_{\mathcal{A}(W')} , \quad (7.27)$$

for all double cone W .⁸ Any normal state on an irreducible DHR representation is called a *DHR state*. The collection of all vector states affiliated to an irreducible DHR representation corresponds a *DHR (superselection) sector*.

The physical meaning of the above definition may be illustrated by the following remark [47]. Let us take a sequence of double cones W_n which exhaust the spacetime in the limit $n \rightarrow \infty$, i.e. $W_n \subset W_{n+1}$ and $\bigcup_{n \in \mathbb{N}} W_n = \mathbb{R}^d$. Then, $\phi \in \mathfrak{S}(\mathcal{A})$ is a DHR state if and only if

$$\left\| (\phi - \omega)|_{\mathcal{A}(W'_n)} \right\| \xrightarrow{n \rightarrow \infty} 0 , \quad (7.28)$$

where ω is the vacuum state. In other words, the state ϕ describes a localized excitation respect to the vacuum in the sense that it cannot be distinguished from it making observations in a region far away from a bounded region.⁹

Before we study the consequences of the definition 7.4, we give the structural assumptions concerning the observable algebra \mathcal{A} , which we need for the subsequent analysis.

Assumption 7.5. The observable algebra is described by a net of C^* -algebras $\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0)$ in the vacuum representation, satisfying all the axioms of definitions 2.56 and 2.58.¹⁰ We also assume the following extra assumptions.

⁸The unitary operator $V_W : \mathcal{H}_0 \leftrightarrow \mathcal{H}_\pi$ that provides the above unitarily equivalence may depends (and in general does) on the region W .

⁹In this case, the charge associated to the state ϕ has to be a global charge, not corresponding to a gauge symmetry, since gauge charges can be measured by the electric flux through a shell of large radius around the charge. Hence the state ϕ can be distinguished from the vacuum in the asymptotic spacelike infinity.

¹⁰This is possible without loss of generality because of the local unitary equivalence of all representations

1. Type of local algebras:

- a) Local algebras $\mathcal{A}(B)$ associated with bounded regions $B \in \mathcal{K}$ are vN algebras, i.e. $\mathcal{A}(B) = \mathcal{A}(B)''$.
- b) Local algebras associated with unbounded regions are C^* -algebras (norm-closed) but they are not, in general, vN algebras (weakly closed). In particular, the global algebra $\mathcal{A} := \mathcal{A}(\mathbb{R}^d)$ is a norm closed, but not a weakly closed, algebra.

2. Additivity:¹¹

- a) For a bounded region $B \in \mathcal{K}$, we have that $\mathcal{A}(B) = (\bigcup_{\alpha} \mathcal{A}(B_{\alpha}))''$ for any causal cover $B := \text{int} \left(D \left(\overline{\bigcup_{\alpha} B_{\alpha}} \right) \right)$.
- b) For an unbounded region $\mathcal{O} \in \mathcal{K}$, we have that $\mathcal{A}(\mathcal{O}) = \overline{\bigcup_{\alpha} \mathcal{A}(D_{\alpha})}^{\|\cdot\|}$ for any causal cover $\mathcal{O} := \text{int} \left(D \left(\overline{\bigcup_{\alpha} \mathcal{O}_{\alpha}} \right) \right)$.¹²

3. Duality for double cones:¹³

$$\mathcal{A}(W) = \mathcal{A}(W')' , \quad \text{for all double cones } W . \quad (7.29)$$

Let us give some comments about these assumptions. Defining the algebras for bounded regions as vN algebras is a useful technical assumption, whereas defining the algebras for unbounded regions just as C^* -algebras, but not vN algebras, is more than a technical assumption. In fact, if W is a double cone, the vN algebra $\mathcal{A}(W)''$ associated with the unbounded region W' is too big enough that it spoils the criterion (7.27). In other words, it can be shown that, in any AQFT and under very general assumptions, there is no representation (beyond the vacuum one) that satisfies definition 7.4 if we replace $\mathcal{A}(W')$ by $\mathcal{A}(W)''$ in (7.27). Moreover, if we assume that the global algebra $\mathcal{A} = \mathcal{A}(\mathbb{R}^d)$ is a vN algebra, then we have that $\mathcal{A}'' = \mathcal{B}(\mathcal{H}_0)$ since the vacuum representation is irreducible (the vacuum state is pure). But, such an algebra has only one sector since it has only one (non-equivalent) irreducible representation, which is given by itself. We will show how these issues appear in the example of the bosonic subnet of the free fermion field introduced above.¹⁴ It is important to emphasize, the physical fact, that the existence of different choices for the algebra of two double cones is independent if the algebras of unbounded

of interest (equation (7.27)).

¹¹This additivity property is equivalent to the one defined in 2' of definition 2.75, but adjusted to the present scenario.

¹²This implies that \mathcal{A} satisfies weak additivity in the sense of definition 2.61.

¹³This version of the duality is weaker than the usual one $\mathcal{A}(W)' = \mathcal{A}(W')$ (condition 4 in definition 2.75). This last one implicitly assumes that the algebra $\mathcal{A}(W')$ is a vN algebra since it is the commutant of the $\mathcal{A}(W)$. In fact, assumption 3 above is equivalent to $\mathcal{A}(W)' = \mathcal{A}(W)''$.

¹⁴Equivalently, we could define all the local algebras to be vN algebras paying the cost of changing the relation (7.27) by

$$\pi|_{\bigcup_{\alpha} \mathcal{A}(W_{\alpha})} \cong \pi_0|_{\bigcup_{\alpha} \mathcal{A}(W_{\alpha})} , \quad (7.30)$$

regions are defined to be vN algebras or not. But, in order to relate this ambiguity, on the choice of algebras, with the existence of SS, we must use the above structural setup. The additivity property is motivated by the example of the previous section. For example, for two spacelike separated double cones W_1, W_2 the vN algebra $\mathcal{A}(W_1) \vee \mathcal{A}(W_2)$ is the smallest vN algebra that we can assign to the region $W_1 \vee W_2$ satisfying isotony, i.e. $\mathcal{A}(W_1), \mathcal{A}(W_2) \subset \mathcal{A}(W_1 \vee W_2)$. Any other choice has to contain more operators, which are not locally generated. The aim here is to clarify which are the other possible choices. The duality condition is a kind of maximality condition for the algebras of complementary regions. In this case, we assume that such a condition holds for double cones, which are the only topological trivial (connected and simply connected) causally complete regions. The physical meaning is that, for topological trivial regions, any operator can be defined locally. This assumption has been proved to hold in many known concrete models of AQFT.

7.1.1.1. Localized endomorphisms

Now we discuss the consequences of definition 7.4. The aim of this subsection is to obtain a clearer way to describe the DHR superselection sectors.

First of all, we see that the vacuum representation $\pi_0(\mathcal{A})$ has been identified with the defining representation \mathcal{A} . In other words, $\pi_0(\mathcal{A}(\mathcal{O})) \equiv \mathcal{A}(\mathcal{O})$. According to (7.27), given a DHR representation π and a double cone W , there exists a unitary operator $V_W : \mathcal{H}_0 \rightarrow \mathcal{H}_\pi$ such that

$$\pi(A) = V_W A V_W^\dagger, \quad \forall A \in \mathcal{A}(W'). \quad (7.31)$$

Now we define $\rho_W : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_0)$ by¹⁵

$$\rho_W(A) := V_W^\dagger \pi(A) V_W, \quad \forall A \in \mathcal{A}, \quad (7.32)$$

which is also a representation of the algebra \mathcal{A} (globally) unitarily equivalent to π , and hence, also satisfies (7.27). Moreover, it can be shown that $\rho_W \in \text{End}(\mathcal{A})$, which, in particular, implies that $\rho_W(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$. According to (7.31) and (7.32), we have that

$$\rho_W(A) = A, \quad \forall A \in \mathcal{A}(W'). \quad (7.33)$$

Any endomorphism satisfying (7.33) is called a *localized endomorphism* (with support in W).

Since the unitary equivalence (7.31) holds not for one but for all double cones, we also have that for any other double cone \tilde{W} there exists a localized endomorphism $\rho_{\tilde{W}}(\cdot) :=$

where the union runs over all the double cones $W_\alpha \subset W'$. However, the global algebra \mathcal{A} cannot be a vN algebra in order to have non-trivial SS.

¹⁵Notice that ρ_W not only depends on W but also on the representation π .

$V_{\tilde{W}}^\dagger \pi(\cdot) V_{\tilde{W}}$ with support in \tilde{W} and a unitary operator $\mathcal{I}_{W\tilde{W}} := V_{\tilde{W}}^\dagger V_W \in \mathcal{A}$ such that

$$\rho_{\tilde{W}}(\cdot) = \mathcal{I}_{W\tilde{W}} \rho_W(\cdot) \mathcal{I}_{W\tilde{W}}^\dagger. \quad (7.34)$$

Any localized endomorphism satisfying (7.34) is called a *localized transportable endomorphism*. The set of localized transportable endomorphism of \mathcal{A} is denoted by Γ , and the subset of Γ containing only irreducible endomorphisms is denoted by Γ_{irr} .

Lemma 7.6. *For any DHR representation π of the observable algebra \mathcal{A} , there exists a transportable localized endomorphism ρ such that $\rho \cong \pi$. And certainly, any localized transportable endomorphism can be considered as a representation of \mathcal{A} satisfying (7.33).*

In other words, the study of DHR representations can be carried out considering localized transportable endomorphisms. The advantage of localized transportable endomorphisms is that they are representations implemented in the vacuum Hilbert space. An endomorphism $\rho \in \Gamma$ may be considered as the physical operation of adding a localized charge over the vacuum state. In fact, the state $\omega_\rho := \omega \circ \rho$ has the property $\omega_\rho(A) = \omega(A)$ for all $A \in \mathcal{A}(W')$. Such a charge could be also transported according to (7.34). Another advantage of considering transportable localized endomorphisms instead of general DHR representations, is that they could be composed. In fact, if $\rho_1, \rho_2 \in \Gamma$ then $\rho_1 \circ \rho_2 \in \Gamma$. This has to be viewed as the physical operation of adding charges on different spacetime regions.

Of course, the correspondence $\pi \mapsto \rho$ of lemma 7.6 is one to many. Moreover, since (globally) unitarily equivalent representations describe the same physics, we must “identify” elements of Γ coming from unitarily equivalent DHR representations. To do that, we define the *inner* or *trivial* elements of Γ as

$$\mathcal{J} := \{ \sigma \in \Gamma : \sigma(A) = UAU^\dagger \text{ with } U \in \mathcal{A}(W) \text{ unitary and } W \text{ a double cone} \}, \quad (7.35)$$

which is a proper subset of the inner automorphisms of \mathcal{A} .¹⁶ Two endomorphisms $\rho_1, \rho_2 \in \Gamma$ are defined to be *equivalent* $\rho_1 \sim \rho_2$ iff there exists $\sigma \in \mathcal{J}$ such that $\rho_2 = \sigma \circ \rho_1$. Then, we have the following lemma.

Lemma 7.7. *Let π_1 and π_2 be (globally) unitarily equivalent DHR representations, and let ρ_1 and $\rho_2 \in \Gamma$ be its corresponding transportable localized endomorphisms. Then $\rho_1 \sim \rho_2$.*

Proof. See [46]. □

According to the above lemma, it is useful to consider the quotient space Γ/\mathcal{J} . The elements of Γ/\mathcal{J} are equivalence classes containing transportable localized endomorphism

¹⁶An automorphism $\sigma \in \text{Aut}(\mathcal{A})$ is called *inner*, if there exists a unitary operator $U \in \mathcal{A}$ such that $\sigma(A) = UAU^\dagger$ for all $A \in \mathcal{A}$.

which are related to each other by inner transportable localized endomorphism. Then, it is not difficult to see that Γ/\mathcal{J} is in one-to-one correspondence with the set of all unitarily inequivalent DHR representations. In other words, the set Γ/\mathcal{J} contains exactly one copy of any unitarily inequivalent DHR representation. Furthermore, according to definition 7.4, $\Gamma_{\text{irr}}/\mathcal{J}$ is in one-to-one correspondence with the DHR superselection sectors.

Many operations could be performed inside the set Γ . For example, given any two endomorphisms $\rho_1, \rho_2 \in \Gamma$ there exists $\rho \in \Gamma$ such that $\rho_1 \oplus \rho_2$ is unitarily equivalent (as a representation) to ρ .¹⁷ We also have that, if $\mathcal{S} \subset \mathcal{H}_0$ is a proper invariant subspace of $\rho \in \Gamma$, then there exists $\tilde{\rho} \in \Gamma$ such that $\rho|_{\mathcal{S}}$ is unitarily equivalent to $\tilde{\rho}$. These operations are the usual operations of taking direct sum or restriction (to invariant subspaces) of representations. Furthermore, as we have claimed above, Γ naturally has a (semi) product operation given by the composition of endomorphisms. In fact, $\Gamma \subset \text{End}(\mathcal{A})$. From the mathematical standpoint, the set Γ , equipped with the three operations above, defines a C^* -tensor category [57]. All the operations are also well-defined in the quotient space Γ/\mathcal{J} .

Since DHR sectors are in one-to-one correspondence with $\Gamma_{\text{irr}}/\mathcal{J}$, any element of Γ_{irr} is called a DHR sector. Because of this, any element of Γ_{irr} is called an (irreducible) DHR sector. Moreover, according to the discussion above, any non-irreducible element of Γ could be formed by taking direct sums of irreducible ones. It is for this reason that, making an abuse of language, any (generally reducible) transportable localized endomorphism is also called a DHR sector. Trivial transportable localized endomorphisms correspond to the trivial (vacuum) sector.

7.1.1.2. Intertwiners and failure of duality

For our purpose, we are interested in two endomorphism $\rho_1, \rho_2 \in \Gamma$ with supports in strictly spacelike separated double cones W_1 and W_2 . Any operator $\mathcal{I}_{12} \in \mathcal{B}(\mathcal{H}_0)$ satisfying

$$\mathcal{I}_{12}\rho_1(A) = \rho_2(A)\mathcal{I}_{12}, \quad \forall A \in \mathcal{A}, \quad (7.36)$$

is called an *intertwiner*. If $\rho_1 \sim \rho_2$, the intertwiner is a unitary operator, but, in general, \mathcal{I}_{12} is simply a partial isometry. When ρ_1 and ρ_2 are disjoint (in particular, when they are unitarily inequivalent irreducible endomorphisms), we have that $\mathcal{I}_{12} = 0$. From (7.36) we have that

$$\mathcal{I}_{12}A = A\mathcal{I}_{12}, \quad \forall A \in \mathcal{A}(W'_3), \quad (7.37)$$

where W_3 is any double cone containing W_1 and W_2 . Then, by the duality condition for double cones, $\mathcal{I}_{12} \in \mathcal{A}(W'_3)' = \mathcal{A}(W_3) \subset \mathcal{A}$, which means that the intertwiner belongs to the observable algebra. Furthermore, relation (7.37) could be strengthened to $\mathcal{I}_{12}A = A\mathcal{I}_{12}$

¹⁷Notice that $\rho_1 \oplus \rho_2$ is a representation in $\mathcal{H}_0 \oplus \mathcal{H}_0$, whereas ρ is a representation in \mathcal{H}_0 .

for all $A \in \mathcal{A}(W'_1) \cap \mathcal{A}(W'_2)$, which means

$$\mathcal{I}_{12} \in (\mathcal{A}(W'_1) \cap \mathcal{A}(W'_2))'. \quad (7.38)$$

Moreover, because of isotony, we have that $\mathcal{A}((W_1 \vee W_2)') \subset \mathcal{A}(W'_1), \mathcal{A}(W'_2)$ which implies $\mathcal{A}((W_1 \vee W_2)') \subset \mathcal{A}(W'_1) \cap \mathcal{A}(W'_2)$. Hence, equation (7.38) implies

$$\mathcal{I}_{12} \in \mathcal{A}((W_1 \vee W_2)')'. \quad (7.39)$$

Formula (7.38) suggests that the existence of intertwiners may contribute to the failure of the duality condition for two double cones. In fact, we have shown that

$$\mathcal{A}(W_1 \vee W_2) \subset \mathcal{A}(W_1 \vee W_2) \vee \{\mathcal{I}_{12}\} \subset \mathcal{A}((W_1 \vee W_2)')', \quad (7.40)$$

where $\{\mathcal{I}_{12}\}$ denotes the set of all intertwiners between W_1 and W_2 . It is important to remark that $\{\mathcal{I}_{12}\}$ includes all the intertwiners between all DHR representations, despite they are reducible or irreducible. However, because we are interested in the algebra generated by such intertwiners plus the additive algebra $\mathcal{A}_{W_1 \vee W_2}$, we need only a “basis” of intertwiners. We will show in section 7.1.3 that such a basis could be constructed using only the irreducible sectors $\rho \in \Gamma_{\text{irr}}$. A precise knowledge of the general structure of the set $\{\mathcal{I}_{12}\}$ will be remarkably useful to compute entanglement measures on the algebra $\mathcal{A}((W_1 \vee W_2)')'$.

Unitary intertwiners can be used to construct localized endomorphisms. Let $\rho \in \Gamma$ with support in W and take a sequence of equivalent morphisms $\rho_\lambda \in \Gamma$ ($\lambda \in \mathbb{R}$) whose respective localization regions move to infinity, so as to become eventually spacelike to any given double cone. In other words, we can consider that ρ_λ has support in $W_\lambda := W + \lambda a$, where $a \in \mathbb{R}^d$ is a fixed spacelike vector. Then, there exists a corresponding sequence of unitary intertwiners $\mathcal{I}_\lambda \in \mathcal{A}((W \vee (W + \lambda a))')'$ such that

$$A = \mathcal{I}_\lambda \rho(A) \mathcal{I}_\lambda^\dagger, \quad \forall A \in \mathcal{A}((W + \lambda a)'). \quad (7.41)$$

In the limit when $\lambda \rightarrow +\infty$ we have that (7.41) holds for all $A \in \mathcal{A}(D)$ where $D \in \mathcal{K}$ is any bounded region. Since any element of global algebra \mathcal{A} is the uniform limit of elements belonging to algebras of bounded regions, we expect that (7.41) holds for all $A \in \mathcal{A}$ in the limit when $\lambda \rightarrow +\infty$. More precisely, it can be shown that [47]

$$\rho(A) = \lim_{\lambda \rightarrow +\infty} \mathcal{I}_\lambda^\dagger A \mathcal{I}_\lambda, \quad (7.42)$$

where this limit has to be understood in the strong operator topology. In other words, the intertwiner operator could be used to create the localized endomorphism. Essentially, the

intertwiner changes the position of the charge, and then, (7.42) represents the physical act of bringing the charge from infinity.

Let $\sigma_i \in \mathcal{J}$ with $\sigma_i(A) := U_i A U_i^\dagger$ with unitaries $U_i \in \mathcal{A}(W_i)$. In this case, we have that the unitary operator $\mathcal{I}_{12} := U_2 U_1^\dagger$ satisfies

$$\mathcal{I}_{12} \sigma_1(A) = \sigma_2(A) \mathcal{I}_{12}, \quad \forall A \in \mathcal{A}, \quad (7.43)$$

i.e. it is an intertwiner. In this case, we have that $\mathcal{I}_{12} \in \mathcal{A}(W_1 \vee W_2)$ and we say that \mathcal{I}_{12} is a *trivial* intertwiner. This happens because \mathcal{I}_{12} intertwines between inner endomorphisms both representing the vacuum sector. However, we expect that intertwiners corresponding to non-trivial sectors do not belong to the additive algebra $\mathcal{A}((W_1 \vee W_2))$, and they produce an explicit breakdown of the duality condition for two double cones. In fact, if we assume that an intertwiner $\mathcal{I}_{12} \in \mathcal{A}(W_1 \vee W_2)$ can be written as $\mathcal{I}_{12} := U_2 U_1^\dagger$ with unitaries $U_j \in \mathcal{A}(W_j)$, then it is not difficult to show that the endomorphisms ρ_j in (7.36) are trivial, $\rho_j(A) = U_j A U_j^\dagger$. In other words, we expect that any non-trivial DHR sector gives place to a non-additive intertwiner.

Conversely, we expect that any operator $\mathcal{I}_{12} \in \mathcal{A}((W_1 \vee W_2)')'$ but $\mathcal{I}_{12} \notin \mathcal{A}((W_1 \vee W_2))$ should correspond to an intertwiner of a non-trivial DHR sector. Despite there is no general proof of this statement, we can argue in favor of that as follows. Starting with such an operator \mathcal{I}_{12} , we may expect that we can use it to construct a family of operators $\mathcal{I}_\lambda \in \mathcal{A}((W_1 \vee (W_2 + \lambda a))')'$ which translates the charge to $W_2 + \lambda a$, such that in the limit $\lambda \rightarrow +\infty$ gives place to localized endomorphism according to (7.42). Furthermore, it is expected that such a localized endomorphism is non-trivial whenever \mathcal{I}_{12} is non-trivial, i.e. $\mathcal{I}_{12} \notin \mathcal{A}((W_1 \vee W_2))$.

In other words, it is expected that

$$\mathcal{A}((W_1 \vee W_2)')' = \mathcal{A}(W_1 \vee W_2) \vee \{\text{DHR intertwiners between } W_1 \text{ and } W_2\}, \quad (7.44)$$

and intertwiners corresponding to non-trivial DHR sectors does not belong to the additive algebra $\mathcal{A}((W_1 \vee W_2))$.

Example: bosonic subnet of the free fermion field

Now, we continue with the example introduced above. The Fock Hilbert space of the free fermion is decomposed as $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where \mathcal{H}_0 (resp. \mathcal{H}_1) is formed by all the vectors with an even (resp. odd) number of fermions. In particular, we recall that the vacuum vector $|0\rangle$ belongs \mathcal{H}_0 . Moreover, the observable algebra \mathcal{A} leaves both subspaces

$\mathcal{H}_0, \mathcal{H}_1$ invariant. In this way, we can identify two representations

$$(\pi_0, \mathcal{H}_0) : \quad \pi_0(A) |\psi_0\rangle := A |\psi_0\rangle, \quad A \in \mathcal{A} \text{ and } |\psi_0\rangle \in \mathcal{H}_0, \quad (7.45)$$

$$(\pi_1, \mathcal{H}_1) : \quad \pi_1(A) |\psi_1\rangle := A |\psi_1\rangle, \quad A \in \mathcal{A} \text{ and } |\psi_1\rangle \in \mathcal{H}_1. \quad (7.46)$$

Despite both representations has the same assignation rule $\pi_k(A) = A$ ($k = 1, 2$), they are not the same representation since they act in different Hilbert spaces. Moreover, both representations are not unitarily equivalent. This is because both are covariant respect to unitarily inequivalent representations of the Poincaré group. In fact, the one acting on \mathcal{H}_0 is a vacuum representation, but the one acting on \mathcal{H}_1 cannot be a vacuum representation since it has not a Poincaré invariant vector.

Let W be a double cone, and consider the fermionic operator

$$V_W := \psi^\dagger(f) + \psi(f), \quad (7.47)$$

where $\text{sup}(f) \subset W$. This operator does not belong to the observable algebra \mathcal{A} , but it belongs to the fermionic algebra \mathfrak{F} . Moreover, $V_W \in \mathfrak{F}(W)$. In addition, we choose $f(x) := \alpha(\bar{x}) \delta(x^0)$ with $\int d^{d-1}x \alpha(\bar{x})^\dagger \alpha(\bar{x}) := 1$.¹⁸ Using the anticommutation relations $\{\psi(0, \bar{x}), \psi^\dagger(0, \bar{y})\} = \delta(\bar{x} - \bar{y})$, it can be shown that $V_W^\dagger = V_W = V_W^{-1}$. We also have that $V_W \mathcal{H}_0 = \mathcal{H}_1$ and $V_W \mathcal{H}_1 = \mathcal{H}_0$. Then, we can consider the operator V_W as acting¹⁹

$$V_W : \mathcal{H}_0 \rightarrow \mathcal{H}_1. \quad (7.48)$$

In this way, V_W is a unitary operator from \mathcal{H}_0 onto \mathcal{H}_1 , and $V_W^\dagger : \mathcal{H}_1 \rightarrow \mathcal{H}_0$. It is not hard to see that

$$V_W^\dagger \pi_1(A) V_W = \pi_0(A), \quad \forall A \in \mathcal{A}(W'). \quad (7.49)$$

Moreover, since the choice of the double cone W in (7.47) is arbitrary, we also have that relation (7.49) holds for any double cone. This shows that π_1 is a DHR representation. The localized endomorphism corresponding to π_1 is defined as

$$\rho_W(A) := V_W^\dagger \pi_1(A) V_W, \quad (7.50)$$

We have that $\rho_W(A) \mathcal{H}_0 \subset \mathcal{H}_0$ for all $A \in \mathcal{A}$, and moreover,

$$\rho_W(A) \subset \pi_0(\mathcal{A}) \subset \mathcal{B}(\mathcal{H}_0), \quad \forall A \in \mathcal{A}. \quad (7.51)$$

¹⁸Here we are smearing in space only. This can be done for a free field (see footnote 2). We could have also chosen spacetime smearing functions as well, at the expense of replacing $\int d^{d-1}x \alpha(\bar{x})^\dagger \alpha(\bar{x}) = 1$ by an integral in d dimensions weighted with the anticommutator distribution.

¹⁹In other words, we take the restriction of V_W to the vacuum subspace \mathcal{H}_0 .

Identifying the observable algebra \mathcal{A} with the vacuum representation $\pi_0(\mathcal{A})$, expressions (7.49) and (7.50) imply that

$$\rho_W(A) = A, \quad \forall A \in \mathcal{A}(W'). \quad (7.52)$$

Now, we can do the same construction for two strictly spacelike separated double cones W_1 and W_2 . Then, we have fermionic unitary operators V_{W_1}, V_{W_2} and localized endomorphisms ρ_1, ρ_2 , according to (7.50). The operator $\mathcal{I}_{12} := V_{W_2}^\dagger V_{W_1}$ satisfies

$$\mathcal{I}_{12}\rho_1(\cdot) = \rho_2(\cdot)\mathcal{I}_{12}, \quad (7.53)$$

i.e. it is an intertwiner between both endomorphisms. This intertwiner belongs to the observable algebra \mathcal{A} , it is made out of products of fermionic operators, one belonging to $\mathfrak{F}(W_1)$ and the other one to $\mathfrak{F}(W_2)$, but the product \mathcal{I}_{12} does not belong to the additive algebra $\mathcal{A}(W_1 \vee W_2)$. This is in accordance with what we have anticipated above. In this way, we have shown explicitly, in this example, that the DHR intertwiners are the operators which break down the duality condition for two double cones.

In this model, there is no another non-equivalent DHR representation beyond π_0 and π_1 . There is no non-zero intertwiner between ρ_W and the “vacuum” endomorphism, which is defined as $\iota(A) := A$ for all $A \in \mathcal{A}$. This is because such representations are disjoint, i.e., they correspond to non-equivalent irreducible sectors. Non-irreducible sectors can be formed taking direct sums of irreducible ones, and their corresponding intertwiners could be generated taking products and sums of irreducible intertwiners and operators of the additive algebra $\mathcal{A}(W_1 \vee W_2)$. In other words, for the purpose of the algebra $\mathcal{A}(W_1 \vee W_2) \vee \{\mathcal{I}_{12}\}$, we lose nothing by only considering irreducible intertwiners. The operator V_W is called a *charge creating operator*. It transforms vectors belonging to different sectors, but it does not belong to the algebra \mathcal{A} .

We notice in this example, that there exists a local QFT $\mathfrak{F}(\mathcal{O})$ containing exactly one copy of each irreducible sector. In fact, the fermionic net $\mathfrak{F}(\mathcal{O})$ is an AQFT satisfying all the axioms of the definition 2.70. This net is covariant respect to the universal covering of the Poincaré group. The observable net is the subnet formed by the bosonic elements. The operator $\Gamma \in \mathcal{B}(\mathcal{H})$ defined by²⁰

$$\Gamma |\psi_0\rangle := |\psi_0\rangle, \quad |\psi_0\rangle \in \mathcal{H}_0, \quad (7.54)$$

$$\Gamma |\psi_1\rangle := -|\psi_1\rangle, \quad |\psi_1\rangle \in \mathcal{H}_1, \quad (7.55)$$

forms a group of unitaries $G := \{\mathbf{1}, \Gamma\}$ isomorphic to \mathbb{Z}_2 .²¹ An operator $A \in \mathfrak{F}$ belongs to

²⁰Do not confuse this operator with the set of localized transportable endomorphisms of \mathcal{A} .

²¹In this case, Γ is also the grading of the field algebra (see section 2.2.4).

the observable algebra if and only if it is G -invariant, i.e. $A \in G'$. This enveloping AQFT is named the “field algebra”, in contrast to \mathcal{A} , which is called the observable algebra.²² Remarkably, this structure is quite general. Given any observable AQFT \mathcal{A} , it always exists a field algebra \mathfrak{F} containing copies of all DHR sectors, and a group of unitaries G such that the observable algebra is formed by the G -invariant elements of \mathfrak{F} . Moreover, the field algebra contains no further SS.

7.1.2. Field algebra

A *field algebra* consists in a net of C^* -algebras $\mathcal{O} \in \mathcal{K} \mapsto \mathfrak{F}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ satisfying all the axioms of definition 2.70, and a compact group G of unitaries, called the *symmetry group*,²³ such that the observable algebra is defined as the G -invariant part of \mathfrak{F} , i.e

$$\mathcal{A} := \mathfrak{F} \cap G'. \quad (7.56)$$

Here, we again define the algebras of bounded regions as vN algebras, whereas the algebras of the unbounded regions are just C^* but not vN algebras. The field algebra is assumed to satisfy twisted duality for double cones. i.e $\mathfrak{F}(W')^{t'} = \mathfrak{F}(W)$.

The symmetry group must commute with the Poincaré unitaries²⁴

$$[U_G(g), U_P(l)] = 0, \quad \forall g \in G, \forall l \in \tilde{\mathcal{P}}_+^\uparrow. \quad (7.57)$$

Because of this, the elements of G preserve the local algebras

$$\alpha_g(\mathfrak{F}(\mathcal{O})) := U(g)\mathfrak{F}(\mathcal{O})U(g)^\dagger = \mathfrak{F}(\mathcal{O}), \quad \forall \mathcal{O} \in \mathcal{K}, \forall g \in G, \quad (7.58)$$

i.e., they represent *internal symmetries*. Moreover, the Poincaré unitary $U_P(z)$ corresponding to the element $z := (0, -\mathbf{1}) \in \mathbb{R}^d \ltimes \tilde{\mathcal{L}}_+^\uparrow$, which is the grading of the field algebra, must belong also to the symmetry group G . Then, the observable algebra \mathcal{A} does not contain fermionic operators.

The local observable algebras are defined as

$$\mathcal{A}(W) := \mathfrak{F}(W) \cap G', \quad \text{if } W \text{ is a double cone,} \quad (7.59)$$

and the algebras for non-double cones are defined in order to satisfy additivity. It can be

²²As we have anticipated in section 2.2.4, the field algebra may contain non-observable quantities which they do not commute at spacelike distance.

²³In the literature of AQFT, G is called the gauge group regardless it represents a group of global symmetries.

²⁴From now on, U denotes the representation of the symmetry group whereas U_P denotes the representation of the Poincaré group.

shown that $\mathfrak{F}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O})'$. The observable net $\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ satisfies all the axioms of definition 2.56.

It is important to emphasize that the representation of the observable algebra \mathcal{A} on the Hilbert space \mathcal{H} is highly reducible. In fact, it can be shown that the Hilbert space \mathcal{H} decomposes as

$$\mathcal{H} := \bigoplus_{\sigma \in \hat{G}} \mathbb{C}^{d_\sigma} \otimes \mathcal{H}_\sigma, \quad (7.60)$$

where the direct sum runs over the spectrum \hat{G} of G , i.e. the set of all non-equivalent unitary irreducible representations. All group representations are finite dimensional since G is compact, and $d_\sigma < \infty$ denotes their dimensions. The observable algebra \mathcal{A} and the symmetry group reduce according to the decomposition (7.60) as

$$U(g) = \bigoplus_{\sigma \in \hat{G}} U_\sigma(g) \otimes \mathbf{1}_{\mathcal{H}_\sigma}, \quad (7.61)$$

$$\mathcal{A} \ni A = \bigoplus_{\sigma \in \hat{G}} \mathbf{1}_{d_\sigma} \otimes \pi_\sigma(A). \quad (7.62)$$

U_σ is the unitary irreducible representation of G of type $\sigma \in \hat{G}$. $\pi_\sigma : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\sigma)$ are inequivalent irreducible representations of the observable algebra \mathcal{A} [45]. Each of them gives place to different AQFTs

$$\mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}_\sigma(\mathcal{O}) := \pi_\sigma(\mathcal{A}(\mathcal{O})) \subset \mathcal{B}(\mathcal{H}_\sigma), \quad (7.63)$$

satisfying all the axioms of definition 2.56. Moreover, the vacuum vector $|0\rangle \in \mathcal{H}$ belongs to the Hilbert space $\mathcal{H}_0 := \mathcal{H}_{\text{triv}}$ corresponding to the trivial representation of G , and its corresponding observable algebra representation $\mathcal{A}_0(\mathcal{O}) := \mathcal{A}_{\text{triv}}(\mathcal{O})$ is a vacuum representation, i.e. it satisfies also all the axioms of definition 2.58.

It is important to remark that all the representations in (7.63) satisfy the DHR criteria (definition 7.27) respect to the vacuum representation $\pi_0 := \pi_{\text{triv}}$. Moreover, the localized endomorphism $\rho_{W,\sigma} \in \text{End}(\mathcal{A})$ corresponding to a one-dimensional representation can be always constructed as in (7.50)

$$\rho_{W,\sigma}(A) := V_{W,\sigma}^\dagger A V_{W,\sigma}, \quad (7.64)$$

where the charge creating operators $V_{W,\sigma} \in \mathfrak{F}$ but $V_{W,\sigma} \notin \mathcal{A}$. For a general irreducible representation, a slightly modification of relation (7.64) holds (see subsection 7.1.3 below). Non-irreducible representations could be constructed in similar fashion invoking property B (see [2, 57, 58]).²⁵

²⁵Property B is a fundamental property concerning the local algebras of QFT which is expected to hold in general because it follows from the physically motivated postulates of causality, spectrum condition and

In conclusion, starting with a field algebra \mathfrak{F} with a compact symmetry group G , and defining the observable algebra \mathcal{A} as the G -invariant part of \mathfrak{F} , then, the observable algebra decomposes into irreducible DHR representations. The vacuum representation appears only once in the above decomposition. As we did above, it is useful to identify the observable algebra with the vacuum representation, which by the way, they are C^* -isomorphic. In fact, for the purpose of computing information/entanglement measures in the vacuum state (or in any another normal state of \mathcal{A}_0), there is no difference in using \mathcal{A} or \mathcal{A}_0 . The GNS-representation of \mathcal{A} and the vacuum state $\omega(\cdot) = \langle 0 | \cdot | 0 \rangle$ is unitarily equivalent to $(\mathcal{A}_0, \mathcal{H}_0)$.

The most remarkable part of this story is that the above procedure can be reversed. If we start with an observable algebra \mathcal{A}_0 in the vacuum representation, it always exists a unique (up to net isomorphisms) field algebra \mathfrak{F} and a compact symmetry group G , such that the G -invariant part of \mathfrak{F} contains copies of all DHR representations of the observable algebra \mathcal{A}_0 . This is the famous Doplicher-Roberts reconstruction theorem [46, 50].²⁶ In this way, we do not lose any generality assuming that the observable algebra is embedded in bigger algebra containing copies of all DHR SS, as we did for the free fermion field. According to our discussion in section 7.1.1.2, we expect that the field algebra satisfies (twisted) duality for two double cones, since it has no further DHR SS.

7.1.3. Intertwiners and twists

Once we have recognized that the DHR intertwiners are the responsible of the failure of the duality for two double cones, we need to better understand their algebraic properties in order to be able to use them for entanglement measures computations. In the end, we want to construct an entropic order parameter which “measures” the difference between the algebras $\mathcal{A}(W_1 \vee W_2)$ and $\mathcal{A}((W_1 \vee W_2)')' = \mathcal{A}((W_1 \vee W_2)) \vee \{\mathcal{I}_{12}\}$.

With this in mind, we start the structural assumptions of the previous section. The observable algebra \mathcal{A} is defined as the G -invariant part of a theory without superselection sectors. This later is field algebra \mathfrak{F} . The neutral charge sector (vacuum representation) \mathcal{A}_0 is embedded in \mathcal{A} . \mathcal{A} and \mathcal{A}_0 are C^* -isomorphic, and they are frequently identified.

In similar way as in relation (7.64), a localized endomorphism associated with a generic (not necessarily irreducible) representation ξ of G localized in a double cone W can be written by means of the *charge creating operators* $V_{W,\xi} \in \mathfrak{F}(W)$. It is possible to choose a family of these operators $\{V_\xi^j \ ;, j = 1, \dots, d_\xi\} \subset \mathfrak{F}(W)$ such that they transform according

weak additivity [57, 152, 153].

²⁶This could be done for spacetime dimensions greater than 2. For $d = 1, 2$ the DHR SS may not come from a compact group, and a slight modification is required.

the unitary representation ξ of the symmetry group²⁷

$$U(g)^\dagger V_\xi^j U(g) = D_\xi(g)_k^j V_\xi^k, \quad g \in G, \quad (7.65)$$

where $D_\xi(g)_k^j$ is the representation matrix. To obtain an endomorphism of \mathcal{A} associated with the representation ξ , we define

$$A \in \mathcal{A} \mapsto \rho_\xi(A) := \sum_{j=1}^{d_\xi} V_\xi^j A V_\xi^{j\dagger}. \quad (7.66)$$

Then, we have that $\rho_\xi(A) \in \mathcal{A}$ because of (7.65). In order to (7.66) defines an endomorphism (respects the product of operators and maps the identity into itself), we need additionally²⁸

$$V_\xi^{j\dagger} V_\xi^k = \delta_{jk} \mathbf{1}_{\mathcal{H}}, \quad (7.67)$$

$$\sum_{j=1}^{d_\xi} V_\xi^j V_\xi^{j\dagger} = \mathbf{1}_{\mathcal{H}}, \quad (7.68)$$

such that V_ξ^j are partial isometries. For the case of one dimensional (irreducible) representations, V_ξ^j is unitary, like in the example of the previous sections.

Acting with these operators in a vector $|\psi_0\rangle \in \mathcal{H}_0$, we obtain vectors $|\psi_\xi^j\rangle = V_\xi^{j\dagger} |\psi_0\rangle$ transforming according to the representation ξ , i.e. $U(g)|\psi_\xi^j\rangle = D_\xi(g)_k^j |\psi_\xi^k\rangle$. In this way, acting with these operators in the vacuum sector \mathcal{H}_0 we are able to construct any vector in the Hilbert space \mathcal{H} , whenever ξ contains copies of all irreducible representations. Furthermore, any operator $F \in \mathfrak{F}$ can be written as

$$F := \sum_{\xi} \sum_{j=1}^{d_\xi} A_{\xi,j} V_\xi^j, \quad (7.69)$$

with $A_{\xi,j} \in \mathcal{A}$, where the representation ξ and the charge creating operators V_ξ^j have to be appropriately chosen depending on F . To write (7.69), we have assumed that every irreducible representation appears at least one representation in the above decomposition.

This endomorphism arises when considering the GNS-representation of the vector state

$$|\psi_\sigma^j\rangle := \sqrt{d_\sigma} V_\sigma^{j\dagger} |0\rangle, \quad (7.70)$$

corresponding to the irreducible representation $\sigma \in \hat{G}$. The factor $\sqrt{d_\sigma}$ is necessary to have

²⁷We do not write the subscript W in order to lighten the notation.

²⁸In particular ρ_ξ is a completely positive map.

a properly normalized state, since²⁹

$$\begin{aligned} \langle 0|V_\sigma^j V_\sigma^{j\dagger}|0\rangle &= \frac{1}{|G|} \sum_{g \in G} \langle 0|U(g)V_\sigma^j U(g)^\dagger U(g)V_\sigma^{j\dagger} U(g)^\dagger|0\rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{j,l=1}^{d_\sigma} D_\sigma(g)_k^j D_\sigma(g)_l^{j*} \langle 0|V_\sigma^k V_\sigma^{l\dagger}|0\rangle = \frac{1}{d_\sigma} \langle 0|\sum_{k=1}^{d_\sigma} V_\sigma^k V_\sigma^{k\dagger}|0\rangle = \frac{1}{d_\sigma}, \end{aligned} \quad (7.71)$$

where we have used the orthogonality relation between irreducible representations,

$$\sum_{g \in G} D_{\sigma_1}(g)_{k_1}^{j_1} D_{\sigma_2}(g)_{k_2}^{j_2*}(g) = \frac{|G|}{d_{\sigma_1}} \delta_{\sigma_1 \sigma_2} \delta_{j_1 j_2} \delta_{k_1 k_2}. \quad (7.72)$$

A same type of algebraic manipulations shows that for any element $A \in \mathcal{A}$

$$\langle \psi_\sigma^j | A | \psi_\sigma^j \rangle = \langle 0 | \rho_\sigma(A) | 0 \rangle = \omega \circ \rho_\sigma(A), \quad (7.73)$$

where ω is the vacuum state and ρ_σ is as in (7.66). In other words, the GNS-representation corresponding to the vector state (7.70) is unitarily equivalent to ρ_σ .

The operators V_ξ^i equipped with the relations (7.67) and (7.68) generates what is called a *Cuntz algebra*. This cannot be represented in finite dimensions. However, the algebra of operators of the form [43]

$$A := \sum_{j,k=1}^{d_\xi} A_{jk} V_\xi^j V_\xi^{k\dagger}, \quad A_{jk} \in \mathbb{C}, \quad (7.74)$$

closes with respect to the matrix multiplication of the coefficients

$$A \cdot B = \sum_{j,k=1}^{d_\xi} \left(\sum_{l=1}^{d_\xi} A_{jl} B_{lk} \right) V_\xi^j V_\xi^{k\dagger}. \quad (7.75)$$

Hence, it is a finite subalgebra of the Cuntz isomorphic to $M_{d_\sigma}(\mathbb{C})$.

If $V_1^j, V_2^j \in \mathfrak{F}$ are charge creating operators localized in two double cones $W_1 \otimes W_2$ associated with the same representation ξ of G , an intertwiner corresponding to this sector can be written as

$$\mathcal{I}_{12}^\xi := \sum_{j=1}^{d_\xi} V_{\xi, W_1}^j V_{\xi, W_2}^{j\dagger}. \quad (7.76)$$

It follows from (7.67) and (7.68) that \mathcal{I}_{12} is unitary. This belongs to observable algebra \mathcal{A} because it is invariant under the symmetry group. Moreover, it commutes will all the

²⁹In the case of infinite compact group, we may replace $\frac{1}{|G|} \sum_{g \in G}$ by $\int_G d\mu(g)$ where $\mu(g)$ is the normalized Haar measure of G .

operators localized outside the two double cones, but it is not generated by \mathcal{A}_{W_1} and \mathcal{A}_{W_2} . Therefore, duality for two double cones does not hold in \mathcal{A} .

The unitary intertwiners (7.76) corresponding to the irreducible representations $\sigma \in \hat{G}$ could be chosen in order they satisfy the character algebra or fusion algebra, i.e.

$$\mathcal{I}_{12}^{\sigma_1} \mathcal{I}_{12}^{\sigma_2} = \sum_{\sigma_3 \in \hat{G}} \eta_{\sigma_1 \sigma_2}^{\sigma_3} \mathcal{I}_{12}^{\sigma_3}, \quad (7.77)$$

where $\eta_{\sigma_1 \sigma_2}^{\sigma_3} \in \mathbb{N}_0$ is the times that the representation σ_3 is included in the decomposition of $\sigma_1 \otimes \sigma_2$.³⁰ We further can chose that

$$\mathcal{I}_{12}^{\sigma \dagger} = \mathcal{I}_{12}^{\bar{\sigma}} \quad \text{and} \quad \mathcal{I}_{12}^0 = \mathbf{1}_{\mathcal{H}}, \quad (7.78)$$

where $\bar{\sigma}$ is the representation conjugate to σ .

The *twist* operators appear in the commutant of the algebra of the two double cones $\mathcal{A}'_{W_1 \vee W_2}$ and they are labeled by elements of the group $g \in G$. They commute with the algebras \mathcal{A}_{W_1} and \mathcal{A}_{W_2} , but they do not commute with \mathfrak{F}_{W_1} (or equivalently \mathfrak{F}_{W_2}). In fact, they can be chosen in a way that they implement the symmetry group in the region W_1 . More precisely, given any double cone $\tilde{W}_1 \supset \bar{W}_1$, there exists $\tau_g \in \mathfrak{F}_{\tilde{W}_1}$ such that

$$\begin{aligned} \tau_g F \tau_g^\dagger &= U(g) F U(g)^\dagger, & \forall F \in \mathfrak{F}_{W_1}, \\ \tau_g F' \tau_g^\dagger &= F', & \forall F' \in \mathfrak{F}_{\tilde{W}'_1}. \end{aligned} \quad (7.79)$$

In particular, on the charge creating operators $V_\xi^j \in \mathfrak{F}_{W_1}$ we have

$$\tau_g^\dagger V_\xi^j \tau_g = \sum_{k=1}^{d_\xi} R_\xi(g)_k^j V_\xi^k. \quad (7.80)$$

Moreover, the twist can be chosen such that they satisfy the group operation [154]³¹

$$\tau_g \tau_h = \tau_{gh} \quad (7.81)$$

$$U(g) \tau_h U(g)^\dagger = \tau_{ghg^{-1}}. \quad (7.82)$$

In other words, the twist operators forms the group algebra of G , which will be denoted by \mathfrak{A}_G .

Twist operators for W'_1 (the complement of a double cone) can be easily defined as $\tau_g := U(g) \tilde{\tau}_g^\dagger$, where $U(g)$ are the global symmetry group transformations and $\tilde{\tau}_g$ are the

³⁰ $\eta_{\sigma_1 \sigma_2}^{\sigma_3} = \langle \chi_{\sigma_3} | \chi_{\sigma_1 \otimes \sigma_2} \rangle_G$ where χ_σ is the character of the representation σ and $\langle \cdot | \cdot \rangle_G$ denotes the invariant and normalized scalar product on the group algebra G .

³¹This is not the case of the simple twist operator (7.15). To construct twists with these special properties one needs to invoke the split property (definition 4.1) [58, 154].

twist operators of double cone W_2 strictly included in W_1 , i.e. $\bar{W}_2 \subset W_1$.

The twist operators are not in general elements of \mathcal{A} , in the case of non-Abelian symmetry groups. We can form (generally non-unitary) elements of \mathcal{A} by taking linear combinations $\tau_c = \sum_{g \in G} c_g \tau_g$, $c_g \in \mathbb{C}$, and imposing

$$U(g)\tau_c U(g)^\dagger = \tau_c. \quad (7.83)$$

Then, the invariant twist operators are naturally associated with the center of the group algebra $\mathcal{Z}(\mathfrak{A}_G)$ where the coefficients $c_g = c_{hgh^{-1}}$ are invariant under conjugation. The dimension of this center d_G is equal to the number of irreducible representations or the number of different conjugacy classes. The group algebra is equivalent to a direct sum of full matrix algebras $\mathfrak{A}_G := \bigoplus_{\sigma \in \hat{G}} M_{d_\sigma}(\mathbb{C})$, where the group elements are represented by with matrices

$$g \in G \mapsto \bigoplus_{\sigma \in \hat{G}} D_\sigma(g) \in \mathfrak{A}_G. \quad (7.84)$$

The center of the group algebra is then clearly spanned by all diagonal matrices, which are linear combinations of the projectors on each irreducible representation. These projectors are precisely³²

$$P_\sigma := \frac{d_\sigma}{|G|} \sum_{g \in G} \chi_\sigma(g)^* \tau_g, \quad (7.86)$$

where $\chi_\sigma(g)$ is the character of the irreducible representation $\sigma \in \hat{G}$. Then, the invariant twists can be written as

$$\tau_c = \sum_{\sigma \in \hat{G}} c_\sigma P_\sigma \in \mathcal{A}, \quad c_\sigma \in \mathbb{C}. \quad (7.87)$$

Remark 7.8. In $d \leq 2$, there is a difference with respect to higher dimensions. For example, for the chiral CFT as in the previous chapter, we can divide the compactified line into four intervals. Let A_1, A_2 be two disjoint intervals and A_3, A_4 the two disjoint intervals forming the complement of $A_1 \vee A_2$ (see section 6.3.2.6). The intertwiner between A_1 and A_2 belongs to $\mathcal{A}'_{A_3 \vee A_4}$, but the twist operator crossing A_3 (or A_4) also belongs to this algebra. Therefore, the number of additional elements in the algebra of $\mathcal{A}'_{A_3 \vee A_4}$ respect to the additive one $\mathcal{A}_{A_1 \vee A_2}$ is larger. The difference with respect to higher dimensions is because the topology of the two intervals and its complement is the same in this case. In higher dimensions, the intertwiner between two double cones W_3 and W_4 , placed inside

³²That these operators are projectors follows from the convolution property of the characters

$$\sum_{g \in G} \chi_{\sigma_1}(g) \chi_{\sigma_2}(hg^{-1}) = \frac{|G|}{d_{\sigma_1}} \delta_{\sigma_1 \sigma_2} \chi_{\sigma_1}(h). \quad (7.85)$$

$(W_1 \vee W_2)'$ is trivially included in the additive algebra $\mathcal{A}_{(W_1 \vee W_2)'}$ because W_3, W_4 can be deformed to a coinciding position without crossing W_1, W_2 . In addition, in $d \leq 2$ the DHR sectors do not necessarily come from a group symmetry, and we have a more general theory of sectors determined by their fusion rules under composition, which replace the decomposition of tensor product of group representations as a sum of irreducible representations. The reasons for this difference with higher dimensions are also related to the complications that appear when analyzing the spin-statistics theorem, since charged operators cannot smoothly interchange its positions without crossing each other (see for example [57]).

7.2. Entropy and DHR sectors

We are interested in the mutual information $I(W_1, W_2)$ between two double cones W_1 and W_2 for the vacuum state ω . We have seen that in the model \mathcal{A} , there are two possible different choices of algebras for the region $W_1 \vee W_2$: one with and one without the intertwiners. The algebra $\mathcal{A}_{W_1 \vee W_2}$ without the intertwiners is additive, and hence, it is the appropriate one to produce the mutual information

$$I_{\mathcal{A}}(W_1, W_2) := I(\mathcal{A}_{W_1}, \mathcal{A}_{W_2}) \quad (7.88)$$

in the observable algebra \mathcal{A} .³³ However, it is of obvious interest to look for an information theoretic quantity that senses the contributions of the intertwiners in the algebra of the union. To start with, the simplest thing to do is to focus on the MI corresponding to the field algebra

$$I_{\mathfrak{F}}(W_1, W_2) := I(\mathfrak{F}_{W_1}, \mathfrak{F}_{W_2}). \quad (7.89)$$

The algebra of $\mathcal{F}(W_1 \vee W_2)$ naturally contains the intertwiners while retaining additivity in \mathfrak{F} . Hence, as an order parameter indicative of the presence of DHR SS in \mathcal{A} we can compute

$$\Delta I := I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2). \quad (7.90)$$

This is always positive by the monotonicity property of the MI (see proposition 3.12). We emphasize that, even if $I_{\mathfrak{F}}(W_1, W_2)$ and ΔI look like quantities that depend on \mathfrak{F} , they are in fact properties of \mathcal{A} itself for G -invariant states, from which \mathfrak{F} can be reconstructed. The ultimate physical reason is that the only non-zero vacuum expectation values in \mathfrak{F} are equal to expectation values in \mathcal{A} . In fact, we will later show, in detail, how both quantities are directly written in terms of the model \mathcal{A} .

³³The above mutual information can be computed indistinctly in the observable algebra \mathcal{A} or in the its vacuum sector \mathcal{A}_0 . Moreover, if for example, we use the Araki formula to compute the above expression, we have to use the vacuum GNS-representation of the algebra $\mathcal{A}(W_1) \vee \mathcal{A}(W_2)$ which is unitarily equivalent to $\mathcal{A}_0(W_1) \vee \mathcal{A}_0(W_2)$.

To put this in a sound ground, we will follow some ideas presented in [62]. We first need to review some quantum information tools that will also be useful in the rest of the chapter. This is done in the next section 7.2.1. Next, in section 7.2.2, we describe the entropic order parameter in terms of a RE, which is determined by intertwiners' expectation values. This allows us to put useful lower bounds. The description of the order parameter in terms of twist expectation values is done in section 7.2.3. This gives us a tool for computing upper bounds on ΔI . We show that the difference of mutual informations saturates to $\log |G|$, where $|G|$ is the number of elements in the symmetry group, in the limit when the two regions touch each other. Twist and intertwiners do not commute and satisfy entropic certainty and uncertainty relations. This is described in section 7.2.4. After that, we make different computations using the main ideas developed in sections 7.2.2 and 7.2.3. We study the case of Lie group symmetries in section 7.2.5, the case of regions with different topologies in section 7.2.6, states with excitations of non-Abelian sectors in section 7.2.7, and the case of spontaneous symmetry breaking in section 7.2.8. Finally, in section 7.2.9, we make some remarks about the special case of $d = 2$.

7.2.1. Entropic order parameter

In order to better understand the difference (7.90), we make use of the material developed in section 3.4. For the present case, the natural conditional expectation from the field algebra to the observable algebra $\varepsilon : \mathfrak{F} \rightarrow \mathcal{A}$ is

$$\varepsilon(F) := \int_G d\mu(g) \alpha_g(F) , \quad (7.91)$$

where $d\mu(g)$ is the normalized Haar measure of the compact group G acting unitarily in \mathfrak{F} , and α_g is defined as in (7.58).³⁴ All the properties stated in definition 3.38 about conditional expectations are easily seen to hold for ε . Essentially, ε takes the G -invariant part of an element of \mathfrak{F} . For the case of the even part of the fermion algebra, a general element $F \in \mathfrak{F}$ is of the form $F = A_0 + A_1\psi$ with $A_0, A_1 \in \mathcal{A}$ and $\psi \in \mathfrak{F}$ is a smeared fermion field appropriately chosen depending on F . Then, $\varepsilon(F) = A_0$. It is important to notice that the global vacuum state $\omega \in \mathfrak{S}(\mathfrak{F})$ is ε -invariant

$$\omega = \omega \circ \varepsilon , \quad (7.93)$$

since the vacuum vector $|0\rangle$ is G -invariant.

³⁴For a finite symmetry group G , (7.91) can be rewritten as

$$\varepsilon(A) = \frac{1}{|G|} \sum_{g \in G} U(g) A U(g)^\dagger . \quad (7.92)$$

We are particularly interested in applying property (3.65) to (7.91), when we restrict the action of the conditional expectation to local subalgebras of \mathfrak{F} . For a double cone W , we have that

$$\varepsilon(\mathfrak{F}_W) = \mathfrak{F}_W \cap G' = \mathcal{A}_W. \quad (7.94)$$

Let us now consider the states $\omega_{\mathfrak{F}} := \omega|_{\mathfrak{F}_W}$ and $\omega_{\mathcal{A}} := \omega|_{\mathcal{A}_W}$, which are the restriction of the vacuum state to the algebras \mathfrak{F}_W and \mathcal{A}_W . Then, if we apply (3.65) to the states $\omega \rightarrow \omega_{\mathfrak{F}}$ and $\phi \rightarrow \omega_{\mathcal{A}} \circ \varepsilon$, we obtain³⁵

$$S_{\mathfrak{F}}(\omega_{\mathfrak{F}} | \omega_{\mathcal{A}} \circ \varepsilon) - S_{\mathcal{A}}(\omega_{\mathfrak{F}}|_{\mathcal{A}_W} | \omega_{\mathcal{A}} \circ \varepsilon|_{\mathcal{A}_W}) = S_{\mathfrak{F}}(\omega_{\mathfrak{F}} | \omega_{\mathfrak{F}} \circ \varepsilon). \quad (7.95)$$

Equation (7.95) is trivial since every term is zero independently. This is an immediate consequence of (7.93), which implies $\omega_{\mathfrak{F}} = \omega_{\mathcal{A}} \circ \varepsilon = \omega_{\mathfrak{F}} \circ \varepsilon$ and $\omega_{\mathfrak{F}}|_{\mathcal{A}_W} = \omega_{\mathcal{A}} \circ \varepsilon|_{\mathcal{A}_W}$.

Now we consider the union of strictly spacelike separated double cones W_1 and W_2 . In this case, we have that

$$\varepsilon(\mathfrak{F}_{W_1 \vee W_2}) = \mathfrak{F}_{W_1 \vee W_2} \cap G' = \mathfrak{F}'_{(W_1 \vee W_2)'} \cap G' = \mathcal{A}'_{(W_1 \vee W_2)'} = \mathcal{A}_{W_1 \vee W_2} \vee \{\mathcal{I}_{12}\}, \quad (7.96)$$

where we have used that the field algebra satisfies (twisted) duality for two double cones since it has no DHR SS. From now on, we denote $\mathfrak{F}_{12} := \mathfrak{F}(W_1 \vee W_2)$ and $\mathfrak{F}_j := \mathfrak{F}(W_j)$ (idem for \mathcal{A}). We apply (3.65) to the case of the algebras $\mathfrak{F}_{12} \cong \mathfrak{F}_1 \hat{\otimes} \mathfrak{F}_2$ and its subalgebra $\mathcal{A}_{12} \cong \mathcal{A}_1 \otimes \mathcal{A}_2$, in order to gain information about the differences of the MIs. Note that \mathfrak{F}_{12} contains the intertwiners that belong to \mathcal{A} on top of the elements of \mathcal{A}_{12} . This last algebra does not contain the intertwiners, that belong to the global neutral algebra but not to the one formed additively in $W_1 \vee W_2$. To exploit this fact, we use the conditional expectation $\varepsilon_{12} = \varepsilon_1 \otimes \varepsilon_2 : \mathfrak{F}_1 \hat{\otimes} \mathfrak{F}_2 \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$

$$\varepsilon_{12}(F_1 \hat{\otimes} F_2) = \varepsilon_1(F_1) \otimes \varepsilon_2(F_2), \quad F_j \in \mathfrak{F}_j, \quad (7.97)$$

where $\varepsilon_j : \mathfrak{F}_j \rightarrow \mathcal{A}_j$ are just the restrictions of (7.91) to \mathfrak{F}_j , i.e. $\varepsilon_j(F_j) = \varepsilon(F_j)$ for all $F_j \in \mathfrak{F}_j$. The equation (7.97) is extended to more general elements by linearity. Notice that in $\varepsilon_{12} = \varepsilon_1 \otimes \varepsilon_2$ the group average (7.91) is done on each factor independently. To do that, we can use the twist operators rather than the global group transformations (see eq. (7.79)).

Let us define the restricted states

$$\omega_{12} := \omega|_{\mathfrak{F}_{12}}, \quad \omega_j := \omega|_{\mathfrak{F}_j}, \quad (7.98)$$

$$\phi_{12} := \omega|_{\mathcal{A}_{12}}, \quad \phi_j := \omega|_{\mathcal{A}_j}. \quad (7.99)$$

³⁵To lighten the notation and when there is no place for confusion, we frequently use the subscripts \mathfrak{F} and \mathcal{A} , in the relative entropies, instead of $\mathfrak{F}(W)$ and $\mathcal{A}(W)$.

The states we choose for using in (3.65) are ω_{12} and $(\phi_1 \otimes \phi_2) \circ \varepsilon_{12}$. We have $(\phi_1 \otimes \phi_2) \circ \varepsilon_{12} = \omega_1 \otimes \omega_2$ because both states are invariant under the group transformations on each region separately, and hence they give the same expectation values for any operator. We also have trivially $\omega_{12}|_{\mathcal{A}_{12}} = \phi_{12}$ and $(\phi_1 \otimes \phi_2) \circ \varepsilon_{12}|_{\mathcal{A}_{12}} = \phi_1 \otimes \phi_2$. Hence, from (3.65) follows

$$S_{\mathfrak{F}}(\omega_{12} | \omega_1 \otimes \omega_2) - S_{\mathcal{A}}(\phi_{12} | \phi_1 \otimes \phi_2) = S_{\mathfrak{F}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12}) . \quad (7.100)$$

The terms on the l.h.s of the above equation are exactly the MIs (7.88) and (7.89). Then, we have that

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) = S_{\mathfrak{F}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12}) . \quad (7.101)$$

The MI $I_{\mathfrak{F}}(W_1, W_2)$ measures the vacuum correlations between regions W_1 and W_2 including the intertwiners, while $I_{\mathcal{A}}(W_1, W_2)$ does not include correlations coming from them. When the regions W_1 and W_2 are near to each other, and the set of intertwiners is finite, there will be plenty of correlations but these will be essentially the same in \mathfrak{F} and \mathcal{A} . The leading divergent terms of the MIs will cancel out and only the effect of the intertwiners will make a difference. On the other hand, $S_{\mathfrak{F}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12})$ is not a MI. This measures the difference between two states on the algebra $\mathfrak{F}_1 \hat{\otimes} \mathfrak{F}_2$: one is the vacuum state ω_{12} and the other is essentially the same state but where the intertwiners have been projected to the neutral algebras on each region independently. Intuitively, the conditional expectation ε_{12} kills the intertwiners by destroying their vacuum correlations. The term $S_{\mathfrak{F}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12})$ is called the *entropic order parameter*.

Now we have all the necessary tools to show that the entropic order parameter $S_{\mathfrak{F}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12})$ and the field MI $I_{\mathfrak{F}}(W_1, W_2)$ are indeed objects that pertain directly to the observable algebra \mathcal{A} . Concerning the entropic order parameter, we first notice that the two states appearing in the RE, namely ω_{12} and $\omega_{12} \circ \varepsilon_{12}$, are invariant under the action of the global symmetry group. Secondly, following expression (7.96), we have that $\varepsilon(\mathfrak{F}_{12}) = \mathcal{A}'_{(12)'}$. Therefore, according to (3.66), we conclude that the order parameter can be computed equivalently as

$$S_{\mathfrak{F}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12}) = S_{\mathcal{A}'_{(12)'}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12}) . \quad (7.102)$$

This formula shows transparently that the order parameter is an intrinsic quantity of the model \mathcal{A} itself. Even more surprisingly, the same is true for the MI $I_{\mathfrak{F}}(W_1, W_2)$. This can be written in the theory \mathcal{A} more succinctly as

$$I_{\mathfrak{F}}(W_1, W_2) = S_{\mathcal{A}'_{(12)'}}(\omega_{12} | (\omega_1 \otimes \omega_2) \circ \varepsilon_{12}) . \quad (7.103)$$

Such a relation follows by applying again the formula (3.65) to the present scenario, which

leads to

$$\begin{aligned} S_{\mathcal{A}'_{(12)'}}(\omega_{12}|(\omega_1 \otimes \omega_2) \circ \varepsilon_{12}) &= S_{\mathcal{A}_{12}}(\omega_{12}|\omega_1 \otimes \omega_2) + S_{\mathcal{A}'_{(12)'}}(\omega_{12}|\omega_{12} \circ \varepsilon_{12}) \\ &= I_{\mathcal{A}}(W_1, W_2) + S_{\mathfrak{F}}(\omega_{12}|\omega_{12} \circ \varepsilon_{12}) = I_{\mathfrak{F}}(W_1, W_2). \end{aligned} \quad (7.104)$$

Although we have defined the conditional expectation by means of the field algebra \mathfrak{F} , the conditional expectation ε_{12} can be defined directly in the algebra \mathcal{A} as well. It is the restriction of the above conditional expectation to the smaller algebra $\mathcal{A}'_{(12)'}$. More quantitatively, the action of ε_{12} in \mathcal{A} can be expressed in the following way. A generic element of \mathcal{A}_{12} can be written $A = \sum_{\sigma \in \hat{G}} A_{\sigma} \mathcal{I}_{12}^{\sigma}$ as an expansion in intertwiners of different irreducible representations and where operators A_{σ} commute with the (invariant) twists in \mathcal{A}_{12} . Then, $\varepsilon_{12}(A) = A_{\text{triv}} =: A_0$. Furthermore, from a conceptual perspective, it is unnecessary to know the structure of SS to construct the conditional expectation ε_{12} . In fact, it can be shown that it always exists a unique conditional expectation for the inclusion of algebras $\mathcal{A}_{12} \subset \mathcal{A}'_{(12)'}$.³⁶

Having shown that even if one computes REs in the field algebra \mathfrak{F} one actually ends up with REs of the invariant algebra \mathcal{A} , it turns out to be technically and conceptually simpler to work with the field algebra \mathfrak{F} , and we will do this in what follows.

Remarks on lattice QFT and EE

We usually are interested to work in the lattice QFT because we work with matrix algebras (or more in general type I vN algebras), and hence, we can use the expressions in terms of density matrices developed in section 3.2. We first notice that lattice models with global symmetries are easily constructed. We have a Hilbert space \mathcal{H}_n for each vertex $n \in \mathbb{Z}^{d-1}$ of the lattice on which there is a faithful representation U_n of the group G . The global Hilbert space is the tensor product $\mathcal{H} := \bigotimes_n \mathcal{H}_n$ and the group acts with the tensor product representation $U := \bigotimes_n U_n$. The algebra $\mathfrak{F} := \mathcal{B}(\mathcal{H})$ is the full algebra of operators in \mathcal{H} , and $\mathcal{A} \subset \mathfrak{F}$ is the subalgebra of invariant operators.

One interest in looking at finite dimensional algebras is the following. One may entertain the idea that, even if the entropies have ambiguities in continuum limit in QFT, the particular difference $S_{\mathfrak{F}}(W) - S_{\mathcal{A}}(W)$, of the complete and the neutral models in the same region and for the vacuum state, could be well-defined in the continuum limit, independently of the chosen lattice regularization and the exact definition of the algebras. In such a case, we could use just this difference as an order parameter. With a focus in investigating this question, we collect some formulas for matrix algebras that will be very useful along this chapter.

³⁶This follows because both algebras \mathcal{A}_{12} and $\mathcal{A}'_{(12)'}$ are factors, and the relative commutant is trivial $\mathcal{A}'_{12} \cap \mathcal{A}'_{(12)'} = \{\lambda \cdot \mathbf{1}\}$ (see [72]).

As we have explained in section 3.4, given an inclusion of finite dimensional algebras $\mathfrak{B} \subset \mathfrak{A}$, there always exists a unique conditional expectation $\varepsilon : \mathfrak{A} \rightarrow \mathfrak{B}$ that preserves the trace. This is not the general case for conditional expectations, but the conditional expectation (7.91) restricted to any local algebra $\mathfrak{F}(\mathcal{O})$ preserves the trace since it is an average over automorphisms. For any state ϕ on \mathfrak{A} , we have that

$$S_{\mathfrak{A}}(\phi|\phi \circ \varepsilon) = S_{\mathfrak{A}}(\phi \circ \varepsilon) - S_{\mathfrak{A}}(\phi). \quad (7.105)$$

To show this we write this RE as $\Delta\langle K \rangle - \Delta S$, where ΔS is the difference of the entropies of the two states, $\Delta\langle K \rangle$ is the difference of the expectation values of the "inner" modular Hamiltonian $K := -\log(\rho_{\phi \circ \varepsilon})$, and $\rho_{\phi \circ \varepsilon} \in \mathfrak{A}$ is the statistical operator corresponding to the state $\phi \circ \varepsilon$. Then, we have that

$$\mathrm{Tr}_{\mathfrak{A}}(\rho_{\phi \circ E} a) = \mathrm{Tr}_{\mathfrak{A}}(\rho_{\phi} \varepsilon(A)) = \mathrm{Tr}_{\mathfrak{A}}(\varepsilon(\rho_{\phi}) \varepsilon(A)) = \mathrm{Tr}_{\mathfrak{A}}(\varepsilon(\rho_{\phi}) A), \quad \forall A \in \mathfrak{A}, \quad (7.106)$$

and we get $\rho_{\phi \circ E} = \varepsilon(\rho_{\phi})$ as an operator in \mathfrak{A} . We then have that $\varepsilon(K) = K$ and hence

$$\mathrm{Tr}_{\mathfrak{A}}(\rho_{\phi} K) = \mathrm{Tr}_{\mathfrak{A}}(\varepsilon(\rho_{\phi} K)) = \mathrm{Tr}_{\mathfrak{A}}(\varepsilon(\rho_{\phi}) K) = \mathrm{Tr}_{\mathfrak{A}}(\rho_{\phi \circ \varepsilon} K). \quad (7.107)$$

Then, it follows that $\Delta\langle K \rangle = 0$ and equation (7.105) holds.

Eq. (7.105) is the difference of entropies on the same algebra \mathfrak{A} , between the state ϕ and the corresponding invariant one $\phi \circ \varepsilon$. In contrast, the quantity $S_{\mathfrak{B}}(W) - S_{\mathfrak{A}}(W)$ refers to an entropy difference between an invariant state in two different algebras. Let ϕ_1 and ϕ_2 be two states on \mathfrak{A} , then we have that³⁷

$$S_{\mathfrak{A}}(\phi_1) = -S_{\mathfrak{A}}(\phi_1|\phi_2) - \mathrm{Tr}_{\mathfrak{A}}(\rho_{\phi_1}^{\mathfrak{A}} \log \rho_{\phi_2}^{\mathfrak{A}}), \quad (7.108)$$

$$S_{\mathfrak{B}}(\phi_1) = -S_{\mathfrak{B}}(\phi_1|\phi_2) - \mathrm{Tr}_{\mathfrak{B}}(\rho_{\phi_1}^{\mathfrak{B}} \log \rho_{\phi_2}^{\mathfrak{B}}). \quad (7.109)$$

Let us now assume that ϕ_1 and ϕ_2 are invariant states under some conditional expectation $\varepsilon : \mathfrak{A} \rightarrow \mathfrak{B}$, i.e. $\phi_j = \phi_j \circ \varepsilon$. Then we have that $S_{\mathfrak{A}}(\phi_1|\phi_2) = S_{\mathfrak{B}}(\phi_1|\phi_2)$, and subtracting the above equations we get

$$S_{\mathfrak{A}}(\phi_1) - S_{\mathfrak{B}}(\phi_1) = \mathrm{Tr}_{\mathfrak{B}}(\rho_{\phi_1}^{\mathfrak{B}} \log \rho_{\phi_2}^{\mathfrak{B}}) - \mathrm{Tr}_{\mathfrak{A}}(\rho_{\phi_1}^{\mathfrak{A}} \log \rho_{\phi_2}^{\mathfrak{A}}) = \langle \log \rho_{\phi_2}^{\mathfrak{B}} \rangle_{\phi_1} - \langle \log \rho_{\phi_2}^{\mathfrak{A}} \rangle_{\phi_2}. \quad (7.110)$$

This holds independently of the invariant state ϕ_2 we have chosen, and it is linear in the state ϕ_1 .

Equation (7.110) shows that expecting $S_{\mathfrak{B}}(W) - S_{\mathfrak{A}}(W)$ to be well-defined is incorrect. In particular, it is not ordered by inclusion as in the case of (7.105). The following example exhibits the problems that may occur.

³⁷In equation (7.109), the restricted state $\phi_j|_{\mathfrak{B}}$ is also denoted by ϕ_j .

Example 7.9. Consider the fermion algebra at one site. It is given by the matrix algebra $\mathfrak{A} := M_2(\mathbb{C})$ with basis elements given by the operators $\{\mathbf{1}, c, c^\dagger, c^\dagger c\}$, where c, c^\dagger are the creation and annihilation operators, and the \mathbb{Z}_2 fermionic symmetry is the symmetry group. For an even state ϕ , such as the vacuum, the entropy in this algebra \mathfrak{A} is equal to the one in the neutral algebra \mathfrak{B} generated by $\{\mathbf{1}, c^\dagger c\}$, i.e. $S_{\mathfrak{A}}(\phi) - S_{\mathfrak{B}}(\phi) = 0$. However, if we choose the algebra $\tilde{\mathfrak{A}}$ generated by $\{\mathbf{1}, c + c^\dagger\}$, the entropy will be $S_{\tilde{\mathfrak{A}}}(\phi) = \log(2)$ for any even state, while the entropy in the even subalgebra $\tilde{\mathfrak{B}} := \{\lambda \cdot \mathbf{1}\}$ is zero. Hence, we have that $S_{\tilde{\mathfrak{A}}}(\phi) - S_{\tilde{\mathfrak{B}}}(\phi) = \log(2)$ for any even state.

The above example illustrates what we expect in a lattice QFT. As we enlarge the algebras to obtain the continuum limit, $S_{\mathfrak{F}}(\omega) - S_{\mathcal{A}}(\omega)$ can be fluctuating depending on the precise details on which the algebras are chosen. This highlights the necessity of using the MI difference (7.90) to get unambiguous results.

7.2.2. Intertwiner version: lower bound

The consequence of expressing the difference of MIs (7.90) as a RE is that we can use monotonicity of the RE to put lower bounds. In particular, to produce a lower bound, we can restrict the states to a subalgebra \mathcal{B}_{12} of \mathfrak{F}_{12} ,

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) \geq S_{\mathcal{B}_{12}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12}). \quad (7.111)$$

Moreover, since the expectation values of the intertwiners are the main difference between these states, we have to find a useful subalgebra \mathcal{B}_{12} that contains the relevant information about the intertwiners.

If we have a finite dimensional \mathcal{B}_{12} (or more generally a type I vN algebra) the l.h.s. of (7.111) can be written in terms of EEs if we further require that the conditional expectation maps the subalgebra into itself, i.e. $\varepsilon_{12}(\mathcal{B}_{12}) \subseteq \mathcal{B}_{12}$. Using (7.105), we get a lower bound given by the difference of entropies

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) \geq S_{\mathcal{B}_{12}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12}) = S_{\mathcal{B}_{12}}(\omega \circ \varepsilon_{12}) - S_{\mathcal{B}_{12}}(\omega). \quad (7.112)$$

The fact that this difference is positive is because the charged operators on W_1 and W_2 can have entanglement in the vacuum state, what is reflected in the expectation values of the intertwiners. This entanglement will count for the entropy of the first state on the r.h.s. of (7.112) but not for the second one.

To improve the lower bound, we can try to maximize the entropy difference over all choices of intertwiner operators, i.e. over all choices of the subalgebra \mathcal{B}_{12} . To see what we can do, let us suppose we have a unitary intertwiner $\mathcal{I}_{12} = V_1 V_2^\dagger$ for an Abelian (one-dimensional) sector with unitaries $V_j \in \mathfrak{F}_j$. A good idea is trying to maximize the expect-

tation value³⁸

$$\langle \mathcal{I}_{12} \rangle = \langle V_1 V_2^\dagger \rangle \rightarrow 1. \quad (7.113)$$

In this case, it could be said that both V_1 and V_2 create essentially the same vector state acting on the vacuum. If $V_2 = V_1^{-1}$ we would get exactly 1, but this is not possible since they belong to local algebras of spacelike separated regions.

Of special interest is the case where the region W_2 is essentially W_1' , and hence both regions cover the full space. This is what we have when we compute the universal regularized EE as we have explained in section 4.2.3. In this case, we will be able to get the maximum value in (7.113). Then, let us think directly in the case $W_2 = W_1'$.³⁹ By using the modular conjugation J associated with the algebra $\mathfrak{F}(W_1)$ and the vacuum state, we are able to convert⁴⁰

$$\langle V_1 V_2^\dagger \rangle = \langle V_1 J \tilde{V}_2 J \rangle, \quad (7.114)$$

with $\tilde{V}_2 := J V_2^\dagger J$ belonging to the algebra $\mathfrak{F}(W_1)$. By modular theory, this is the same as

$$\langle V_1 V_2^\dagger \rangle = \langle V_1 \Delta^{\frac{1}{2}} \tilde{V}_2^\dagger \rangle, \quad (7.115)$$

where $\Delta \geq 0$ is the modular operator. Using Schwarz inequality, we get

$$|\langle V_1 V_2^\dagger \rangle|^2 = |\langle V_1 \Delta^{\frac{1}{2}} \tilde{V}_2^\dagger \rangle|^2 \leq \langle V_1 \Delta^{\frac{1}{2}} V_1^\dagger \rangle \langle \tilde{V}_2 \Delta^{\frac{1}{2}} \tilde{V}_2^\dagger \rangle. \quad (7.116)$$

Therefore, to maximize the expectation value we can choose either $V_1 J V_1 J$ or $\tilde{V}_2 J \tilde{V}_2 J$ as intertwiners. Without loss of generality, we take

$$V_2^\dagger := J V_1 J. \quad (7.117)$$

Note that V_1 and V_2^\dagger will be formed by representations of opposite charge because of the action of J , and this is exactly what we need to produce an intertwiner.

Therefore, we need to maximize $\langle V_1 \Delta^{\frac{1}{2}} V_1^\dagger \rangle$ along all possible charge creating operators belonging to $\mathfrak{F}(W_1)$. If we could choose V_1 such as it commutes with the modular operator Δ , then we would get the desired result $\langle V_1 V_2^\dagger \rangle = \langle V_1 V_1^\dagger \rangle = 1$ because of $\Delta|0\rangle = |0\rangle$. Intuitively, this commutation can be achieved by writing V_1 in the basis that diagonalizes the modular Hamiltonian. We can always write a unitary operator that commutes with the density matrix by choosing phases in this basis. However, this unitary will have zero charge because, roughly speaking, the vacuum density matrix commutes with the charge

³⁸As we did before, $\langle F \rangle := \omega(F)$ denotes the vacuum expectation value of an operator $F \in \mathfrak{F}$.

³⁹In this case, the algebras of interest are the vN algebra $\mathfrak{F}(W_1)$ and the vN algebra $\mathfrak{F}(W_1)'' \equiv \mathfrak{F}(W_1)'^t$ generated by $\mathfrak{F}(W_1')$.

⁴⁰In the case of fermionic algebras, the modular theory requires a slightly modification in order to take into account the twisted duality (see for example [155]). However, here we use the usual approach explained in section 3.3.1 because, in the end, we obtain the same result.

operator. Hence, we can solve the problem only in an approximate way, choosing charge creating operators corresponding to modes of the modular Hamiltonian with modular energy tending to zero, or as much invariant under the modular flow as possible. In QFT, we can always approach $\langle V_1 V_2^\dagger \rangle \rightarrow 1$ as much as we want for complementary regions (in many different ways) since $0 \in \mathbb{R}$ is included in the spectrum of the modular Hamiltonian, which is continuous in \mathbb{R} . In section 7.3, we will explore the physical content of this requirement developing some explicit examples.

Notice that this cannot be done if W_1 and W_2 are at a finite distance, since, in that case, J would map the algebra $\mathfrak{F}(W_1)$ into the algebra of the region $W'_1 \supsetneq W_2$. Then, in general, the maximal correlator cannot be achieved exactly for non-zero distance. However, if the regions touch each to the other along some part of their entanglement surfaces, no matter how small is this part, we can think in putting highly localized excitations very near this region of the boundary, where the modular energy is small. In this region, we can think the states are similar to the case where the full region W'_1 is covered by W_2 . For any such a case, we expect that the maximal correlation can be achieved for a convenient choice of excitations approaching the entanglement surface.

Now, coming back to the bound (7.112) on the MI difference, we can have a universal bound for a finite group G when the spacelike separated double cones W_1 and W_2 touch each to the other along a $d - 2$ dimensional piece of their entanglement surfaces. In this case, we expect that we can maximize the value of the intertwiner expectation values. We will see this bound depends only on the number of elements $|G|$ of the symmetry group G . To see this, let us think we have finite subalgebras of operators on each region $\mathcal{B}_j \subset \mathfrak{F}(W_j)$, which are isomorphic to the matrix algebra $\mathcal{B}_1, \mathcal{B}_2 \cong M_N(\mathbb{C})$. We further require that these algebras are transformed into itself by the global group transformations. Let us call P_{ij}^1 and P_{ij}^2 the operators forming the matrix canonical basis of the algebras \mathcal{B}_1 and \mathcal{B}_2 , i.e.

$$P_{ij}^1 P_{kl}^1 = \delta_{jk} P_{il}^1, \quad P_{ij}^{1\dagger} = P_{ji}^1, \quad \sum_{i=1}^N P_{ii}^1 = \mathbf{1}, \quad (7.118)$$

and analogously for W_2 . One way to generate these finite algebras is to use the charge generating operators V_ξ^i for some representation (not necessarily irreducible). In general, they generate an infinite dimensional algebra. However, the finite dimensional algebra (discussed in section 7.1.3) formed by the operators

$$A = \sum_{i,j=1}^{d_\xi} A_{ij} V_\xi^i V_\xi^{j\dagger}, \quad A_{ij} \in \mathbb{C}, \quad V_\xi^i \in \mathfrak{F}(W_1), \quad (7.119)$$

form a matrix algebra isomorphic to $M_{d_\xi}(\mathbb{C})$. However, one can produce a subalgebra without worrying about the partial isometries V_ξ^i . We will give some examples in the next

section.

Now we want to maximize the entanglement between these two algebras. We choose $P_{ij}^1 := JP_{ij}^2J$ and we think that these operators approximately commute with the modular operator. Under this choice, we notice that if $\mathbf{D}_1(g)$ is the unitary matrix representation of the group G in the algebra \mathcal{B}_1 , then $\mathbf{D}_2(g) := \mathbf{D}_1(g)^*$ is the representation of G in the algebra \mathcal{B}_2 . The statistical operator (density matrix) of the vacuum state ω on the algebra $\mathcal{B}_{12} := \mathcal{B}_1 \otimes \mathcal{B}_2$ is

$$\rho_{jl,ik}^\omega := \langle P_{ij}^1 P_{kl}^2 \rangle = \langle P_{ij}^1 J P_{kl}^1 J \rangle = \langle P_{ij}^1 \Delta^{\frac{1}{2}} P_{lk}^1 \rangle \simeq \langle P_{ij}^1 P_{lk}^1 \rangle = \delta_{jl} \langle P_{ik}^1 \rangle. \tag{7.120}$$

Under these assumptions for the state, hermiticity of $\rho_{jl,ik}^\omega$ implies that

$$\langle P_{ik}^1 \rangle = \frac{1}{N} \delta_{ik}, \tag{7.121}$$

$$\rho_{jl,ik}^\omega = \frac{1}{N} \delta_{ik} \delta_{jl}. \tag{7.122}$$

This state is invariant under the adjoint action by any unitary matrix of the form $\mathbf{U} \otimes \mathbf{U}^*$, and in particular, it is invariant under global group transformations that have this form given our choice of algebras. Then, $\omega|_{\mathcal{B}_{12}}$ is a pure state on \mathcal{B}_{12} , and hence we get

$$S_{\mathcal{B}_{12}}(\omega) = 0. \tag{7.123}$$

This means that ω is maximally entangled between \mathcal{B}_1 and \mathcal{B}_2 as expected.

In order to compute the other state $\phi := \omega \circ \varepsilon_{12}$, we need to know how the group acts on each of the algebras. Let us decompose the action of the group on each algebra \mathcal{B}_1 and \mathcal{B}_2 into irreducible representations. Suppose we have irreducible representations $\sigma \in \hat{G}$ of dimension d_σ with multiplicity n_σ . Then, we have that

$$\sum_{\sigma \in \hat{G}} n_\sigma d_\sigma = N. \tag{7.124}$$

Without loss of generality we take the basis vectors that decompose the group representation into irreducible ones and rename the indices of the basis elements as $i \rightarrow (\sigma, s, l)$, where $s = 1 \dots, n_\sigma$ and $l = 1, \dots, d_\sigma$. The state $\phi = \omega \circ \varepsilon_{12}$ is represented by the density matrix

$$\begin{aligned} \rho_{(\sigma_1 s_1 l_1)(\sigma_2 s_2 l_2), (\sigma_3 s_3 l_3)(\sigma_4 s_4 l_4)}^\phi &= \frac{1}{|G|^2} \sum_{g_1, g_2 \in G} D_{\sigma_1}(g_1)_{l_1}^{l_1} D_{\sigma_2}(g_2)_{l_2}^{l_2*} \rho_{(\sigma_1, s_1, l_1)(\sigma_2, s_2, l_2), (\sigma_3, s_3, l_3)(\sigma_4, s_4, l_4)}^\omega D_{\sigma_3}(g_1)_{l_3}^{l_3*} D_{\sigma_4}(g_2)_{l_4}^{l_4} \\ &= \frac{1}{d_{\sigma_1} N} \delta_{\sigma_1 \sigma_2} \delta_{\sigma_2 \sigma_3} \delta_{\sigma_3 \sigma_4} \delta_{s_1 s_2} \delta_{s_3 s_4} \delta_{l_1 l_3} \delta_{l_2 l_4}. \end{aligned} \tag{7.125}$$

In the last equation, we have used the orthogonality relations for irreducible representations

$$\sum_{g \in G} D_{\sigma_1}(g)_{l_2}^{l_1} D_{\sigma_2}(g)_{l_4}^{l_3*} = \frac{|G|}{d_{\sigma_1}} \delta_{\sigma_1, \sigma_2} \delta_{l_1 l_3} \delta_{l_2 l_4}, \quad (7.126)$$

and the formula (7.122). Therefore, the non-zero part of the density matrix has the structure of a direct sum of blocks labelled by the irreducible representations. The density matrix is then

$$\rho^\phi = \bigoplus_{\sigma \in \hat{G}} \left[\frac{n_\sigma d_\sigma}{N} \left(\frac{1}{n_\sigma} \mathbf{J}_{n_\sigma} \oplus \mathbf{0}_{n_\sigma^2 - n_\sigma} \right) \otimes \left(\frac{1}{d_\sigma^2} \mathbf{1}_{d_\sigma^2} \right) \right], \quad (7.127)$$

where \mathbf{J}_n (resp. $\mathbf{0}_n$) denotes the n -dimensional square matrix with all entries equal to 1 (resp. 0). The second factor of (7.127) is proportional to the identity matrix. Both factors in (7.127) are normalized to have unit trace. Hence, writing the fraction of basis vectors with representation $\sigma \in \hat{G}$ as

$$q_\sigma := \frac{n_\sigma d_\sigma}{N}, \quad \sum_{\sigma \in \hat{G}} q_\sigma = 1, \quad (7.128)$$

the vN entropy is

$$S_{\mathcal{B}_{12}}(\phi) = - \sum_{\sigma \in \hat{G}} q_\sigma \log q_\sigma + \sum_{\sigma \in \hat{G}} q_\sigma \log d_\sigma^2. \quad (7.129)$$

We can vary the frequency q_σ of the representation σ in order to achieve maximal entropy difference $S_{\mathcal{B}_{12}}(\phi) - S_{\mathcal{B}_{12}}(\omega) = S_{\mathcal{B}_{12}}(\phi)$, taking into account the constraint (7.128). We get that the maximum is achieved for

$$q_\sigma := \frac{d_\sigma^2}{|G|}, \quad (7.130)$$

where we have used the relation $|G| = \sum_{\sigma \in \hat{G}} d_\sigma^2$ valid for finite groups. This implies that

$$n_\sigma = d_\sigma \frac{N}{|G|}, \quad (7.131)$$

and from (7.129) we arrive to

$$S_{\mathcal{B}_{12}}(\omega \circ \varepsilon_{12}) - S_{\mathcal{B}_{12}}(\omega) = \log |G|. \quad (7.132)$$

Therefore, the optimal multiplicity of a representation is proportional to the dimension of the representation. This is exactly the case of the regular representation of the group. Then, the optimal representation consists of any number of copies of the regular one. Other representations will give weaker constraints. Notice that there is no increase in the entropy by arbitrarily multiplying the representations and enlarging the algebra. The conditional expectation will take into account that redundant copies are not measuring

any new difference between the models since they are produced by the neutral algebra.⁴¹

For the regular representation we have the best lower bound for complementary regions

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) \geq \log |G|. \quad (7.133)$$

As we will see in the next subsection, $\log |G|$ is also an upper bound for the difference of the mutual informations.

In appendix D, we show formally that the regular representation can always be achieved using the charge generators V_σ^i of all irreducible representations. But, from a physical standpoint, we remark that the regular representation is naturally constructed with high frequency by fusion. We will use this idea in the example in section 7.3.2. The reason is that the character of the regular representation is $\chi_{\text{reg}}(g) = |G|\delta_{g,1}$ and then, the regular representation itself is stable under fusion, i.e $\text{reg} \otimes \text{reg} \cong G \text{reg}$. The tensor product of a regular representation with another representation of dimension d_ξ has character $\chi_{\text{reg} \otimes \xi}(g) = d_\xi |G|\delta_{g,1}$, and then it decomposes into exactly d_ξ copies of the regular representation. This is not the case of other representations. For any representation ξ of dimension $d_\xi > 1$, its character satisfies

$$\frac{\chi_\xi(1)}{d_\xi} = 1, \quad \left| \frac{\chi_\xi(g)}{d_\xi} \right| < 1, \quad (7.134)$$

and then, for the product $r = r_1 \otimes r_2$ of two representations

$$\frac{\chi_r(1)}{d_1 d_2} = 1, \quad \left| \frac{\chi_{12}(g)}{d_1 d_2} \right| = \left| \frac{\chi_1(g)}{d_1} \right| \left| \frac{\chi_2(g)}{d_2} \right|, \quad (7.135)$$

i.e. the normalized character always approaches to the one of the regular representation.

Another way to see this is to realize that the tensor product of arbitrary representations R with some fix representation R_0 can be thought of as a stochastic process in the space of the probabilities q_σ with $\sigma \in \hat{G}$. In fact, the new representation $R' := R_0 \otimes R$ will have

$$q_\sigma^{R'} = \sum_{\nu \in \hat{G}} M_{\sigma\nu}^{R_0} q_\nu^R, \quad (7.136)$$

where

$$M_{\sigma\nu}^{R_0} = \sum_{\xi \in \hat{G}} \frac{N_{\xi\nu}^\sigma}{d_\xi d_\nu} d_\sigma q_\xi^{R_0}, \quad (7.137)$$

⁴¹It is interesting to consider differences of Rényi entropies of the state (7.127) of the intertwiner algebra and the vacuum state. These differences of Rényi entropies are all equal to the same constant $\log |G|$ when taking the regular representation and in this limit of maximal entanglement. This feature of a state is named “flat spectrum” in the literature. Presumably, this leads to a flat spectrum of the difference of Rényi MIs between the two models in the limit of touching regions.

and $N_{\xi\nu}^\sigma$ is the fusion matrix giving the number of irreducible representations of type σ that appear in the tensor product of representations ξ and ν . The matrix M^{R_0} is stochastic and it represents a stochastic process since it has positive entries and $\sum_{\sigma \in \hat{G}} M_{\sigma\nu}^{R_0} = 1$. Since for any fixed ξ we have $\sum_{\nu \in \hat{G}} N_{\xi\nu}^\sigma d_\nu \sim d_\sigma$.⁴² It follows that the probability vector $q_\sigma = \frac{d_\sigma^2}{|G|}$ is the fixed point of the stochastic process, i.e., an eigenvector of M^{R_0} of eigenvalue 1. As for any stochastic process, applying it repeatedly will approach the fixed point exponentially fast.

Roughly speaking, the infinite algebra of QFT in a region is formed by infinitely many products of subalgebras and the group representation is closed under fusion. Hence, the frequency of each irreducible representation must be that of the regular representation. In the regular representation, the basis elements are treated on equal footing by the group transformations, and the subspace of the irreducible representation σ has dimension d_σ^2 . Then, the probability of each irreducible sector in vacuum must be given by (7.130).

7.2.3. Twist version: upper bound

The simplest upper bound for ΔI uses the convexity property of the RE (5 in proposition 3.10). To use this property in the present context, we note that

$$\omega_{12} \circ \varepsilon_{12} = \frac{1}{|G|^2} \sum_{g_1 \in G_1, g'_2 \in G_2} \omega_{g_1 g'_2} = \frac{1}{|G|} \sum_{g_1 \in G_1} \omega_{g_1}, \quad (7.138)$$

where $\omega_{g_1 g'_2}(\cdot) := \omega\left(\tau_g^{1\dagger} \tau_{g'}^{2\dagger} \cdot \tau_g^1 \tau_{g'}^2\right)$ and τ_g^j are the twist operators of the regions W_j ($j = 1, 2$), which they commute between each other. In expression (7.138), the group transformations act on the two regions W_1 and W_2 independently. In the second equality we have used the invariance of ω under global group transformations, which implies that $\omega_{g_1 g_2} = \omega_{g_1}$. Using the convexity property we get

$$\sum_{g_1 \in G_1} \frac{1}{|G|} S_{\mathfrak{F}}(\omega_{g_1} | \omega_{12} \circ \varepsilon_{12}) - S_{\mathfrak{F}}\left(\frac{1}{|G|} \sum_{g_1 \in G_1} \omega_{g_1} | \omega_{12} \circ \varepsilon_{12}\right) \leq - \sum_{g \in G} \frac{1}{|G|} \log\left(\frac{1}{|G|}\right) = \log |G|. \quad (7.139)$$

The second RE in (7.139) vanishes while the REs

$$S_{\mathfrak{F}}(\omega_{g_1} | \omega_{12} \circ \varepsilon_{12}) = S_{\mathfrak{F}}\left(\omega_{g_1} | \frac{1}{|G|} \sum_{g_1 \in G_1} \omega_{g_1}\right) \quad (7.140)$$

⁴²This follows from the fact that the tensor product of the regular representation with any other representation is proportional to the regular representation.

for different $g_1 \in G_1$, are all equal because we can transform any one into any other by a group automorphism. Therefore, we get the upper bound⁴³

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) = S_{\mathfrak{F}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12}) \leq \log |G|, \quad (7.141)$$

which together with the lower bound of the previous section, allows us to conclude that, as the two entanglement surfaces touch each other, the bound becomes saturated for a finite group G

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) = \log |G|. \quad (7.142)$$

Defining the *quantum dimension* \mathcal{D} by $\mathcal{D}^2 := \sum_{\sigma \in \hat{G}} d_{\sigma}^2 = |G|$, we can also write this result in the form

$$\Delta I = \log(\mathcal{D}^2), \quad (7.143)$$

and for the regularized entropy

$$\Delta S = \frac{\Delta I}{2} = \log(\mathcal{D}). \quad (7.144)$$

Written in this way, the contribution coincides with the formula for the topological entanglement entropy [3, 4].

It is interesting to note that (7.142) is a purely topological contribution and does not depend on the interactions or whether the models are massive or massless. Of course, the size $\epsilon := \text{dist}(\gamma_{W_1}, \gamma_{W_2})$ between the entanglement surfaces of W_1 and W_2 where saturation is achieved, depends on the typical size where the intertwiners have appreciable expectation values. For a conformal theory and two double cones, ΔI will be a function of the cross-ratio determining the geometry, while for a massive theory we need to cross the scale of the energy gap to see some difference between the MIs to arise, independently of the size of the regions W_1, W_2 .

Moreover, as we have explained above, the order parameter $S_{\mathfrak{F}}(\omega_{12} | \omega_{12} \circ \varepsilon_{12})$ could be in principle computed in the observable algebra \mathcal{A} itself without further necessity of the field algebra nor of the knowledge of the structure of the DHR SS. At the end, if for a given model we were able to compute it in the limit when both regions touch each other, we would know if the underlying model contains DHR SS and which is the size $|G|$ of the symmetry group G . This gives a novel connection between SS and EE.

According to the derivation of (7.139), saturation is only possible if the supports for the states ω_g become disjoint for different $g \in G$. This requires the vacuum expectation values for the squeezed twists that implement group operations in W_1 and not in W_2 to go

⁴³This upper bound might be considered as an intertwiner or twist upper bound, depending on the focus one is taking. But this bound is not tight in general. The tightest upper bound, which we will derive below, comes from analyzing the problem from a twist perspective.

to zero in this limit. We will see later this is also implied by uncertainty relations between twist and intertwiners that do not commute with each other.

An improved upper bound can be obtained by considering the dual version of (7.101) where the RE is based on the complementary algebra of the two regions $S := (W_1 \vee W_2)'$, which is called the *shell*.⁴⁴ This requires a more specific property that we could not find in the mathematical literature. We are proving this property in the lattice and taking the continuum limit afterward.

We again consider the algebra $\mathfrak{F}_{12} \cong \mathfrak{F}_1 \hat{\otimes} \mathfrak{F}_2$ and we call $\mathfrak{F}_S := \mathfrak{F}''_{12'} = \mathfrak{F}^{t'}_{12}$.⁴⁵ The local algebras \mathfrak{F}_1 , \mathfrak{F}_2 and \mathfrak{F}_S commute with each other, and since $W_1 \vee W_2 \vee S$ fulfills the whole spacetime \mathbb{R}^d , then we have that

$$\mathfrak{F}_1 \vee \mathfrak{F}_2 \vee \mathfrak{F}_S = \mathcal{B}(\mathcal{H}) . \quad (7.145)$$

The twist operators on the region W_1 , which form the group algebra, are denoted by G_τ . The invariant part of \mathfrak{F}_{12} under G_τ is $\mathcal{A}_1 \vee \mathfrak{F}_2 \cong \mathcal{A}_1 \otimes \mathfrak{F}_2$. The commutant of this algebra is $(\mathcal{A}_1 \vee \mathfrak{F}_2)' = \mathfrak{F}_S \vee G_\tau$. We have two conditional expectations. The first one is

$$\varepsilon_1 : \mathfrak{F}_1 \hat{\otimes} \mathfrak{F}_2 \rightarrow \mathcal{A}_1 \otimes \mathfrak{F}_2 \quad (7.146)$$

which follows as (7.91) but where now we act with the twists $\tau_g \in G_\tau$ in region W_1 . The “dual” conditional expectation maps

$$\varepsilon_\tau : \mathfrak{F}_S \vee G_\tau \rightarrow \mathfrak{F}_S . \quad (7.147)$$

To describe the action of ε_τ we note that any element $A \in \mathfrak{F}_S \vee G_\tau$ can be uniquely written as $A = \sum_{g \in G} A_g \tau_g$, where the $A_g \in \mathfrak{F}_S$. Then, we take

$$\varepsilon_\tau(A) := A_1 . \quad (7.148)$$

The conditional expectation ε_τ can be obtained directly with the help of the charge creating operators $V_{\text{reg}}^h \in \mathfrak{F}(W_1)$, corresponding to the regular representation, in the following way

$$\begin{aligned} \frac{1}{|G|} \sum_{h \in G} V_{\text{reg}}^{h\dagger} A V_{\text{reg}}^h &= \frac{1}{|G|} \sum_{h \in G} \sum_{g \in G} A_g V_{\text{reg}}^{h\dagger} \tau_g V_{\text{reg}}^h \\ &= \frac{1}{|G|} \sum_{h \in G} \sum_{g \in G} A_g V_{\text{reg}}^{h\dagger} V_{\text{reg}}^{gh} \tau_g = A_1 = \varepsilon_\tau(A) . \end{aligned} \quad (7.149)$$

⁴⁴The term shell is motivated in the case when we want to compute the regularized entropy for a double cone as we have explained in section 4.2.3.

⁴⁵For the purpose of computing REs, we need that all algebras be vN algebras. Hence the local algebras of unbounded regions must be replaced by its double commutant. For example, the field algebra of the shell $\mathfrak{F}_S = \mathfrak{F}_{12'}$ must be replaced by $\mathfrak{F}''_{12'}$, and this later is equal to $\mathfrak{F}^{t'}_{12}$ because of the twist duality.

Furthermore, the definition of $\mathcal{F}_S \vee G_\tau$ and ε_τ does not depend on the precise form of the twists chosen. Moreover, in the continuum QFT, it can be shown that (7.148) is the unique conditional expectation for the inclusion of algebras $\mathfrak{F}_S \subset \mathfrak{F}_S \vee G_\tau$.⁴⁶

For simplicity we take W_1 and W_2 to be two disjoint sets of vertices on the lattice and take \mathfrak{F}_1 and \mathfrak{F}_2 as the algebras of all lattice operators at these vertices. These algebras are in tensor product with the rest of the lattice operators, which means

$$\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_S, \quad (7.150)$$

$$\mathcal{B}(\mathcal{H}) := \mathfrak{F}_1 \hat{\otimes} \mathfrak{F}_2 \hat{\otimes} \mathfrak{F}_S, \quad \mathfrak{F}_j := \mathcal{B}(\mathcal{H}_j), \quad j = 1, 2, S. \quad (7.151)$$

In this setting, we can choose G_τ as the elements of the group acting on the vertices of W_1 , such that G_τ commutes with \mathfrak{F}_S . In this way, we define

$$\tau_g := \bigotimes_{n \in W_1} U_n(g). \quad (7.152)$$

Then, we have that $G_\tau \subset \mathfrak{F}_1$ and $\mathfrak{F}_S \vee G_\tau = \mathfrak{F}_S \otimes G_\tau$.

Because of the invariance of the global vacuum state, we have as in (7.138)

$$S_{\mathfrak{F}_{12}}(\omega | \omega \circ \varepsilon_{12}) = S_{\mathfrak{F}_{12}}(\omega | \omega \circ \varepsilon_1). \quad (7.153)$$

In this lattice setting, using (7.105) we get

$$S_{\mathfrak{F}_{12}}(\omega | \omega \circ \varepsilon_1) = S_{\mathfrak{F}_{12}}(\omega \circ \varepsilon_1) - S_{\mathfrak{F}_{12}}(\omega). \quad (7.154)$$

Since the global vacuum state ω is pure, we can use lemma 3.7 twice and transform (7.154) successively as

$$\begin{aligned} S_{\mathfrak{F}_{12}}(\omega | \omega \circ \varepsilon_1) &= S_{\mathfrak{F}_{12}}(\omega \circ \varepsilon_1) - S_{\mathfrak{F}_S}(\omega) \\ &= S_{\mathcal{A}_1 \otimes \mathfrak{F}_2}(\omega) - S_{\mathfrak{F}_S}(\omega) + (S_{\mathfrak{F}_{12}}(\omega \circ \varepsilon_1) - S_{\mathcal{A}_1 \otimes \mathfrak{F}_2}(\omega)) \\ &= S_{\mathfrak{F}_S \vee G_\tau}(\omega) - S_{\mathfrak{F}_S}(\omega) + (S_{\mathfrak{F}_{12}}(\omega \circ \varepsilon_1) - S_{\mathcal{A}_1 \otimes \mathfrak{F}_2}(\omega)) \\ &= S_{\mathfrak{F}_S \vee G_\tau}(\omega) - S_{\mathfrak{F}_S \vee G_\tau}(\omega \circ \varepsilon_\tau) \\ &\quad + (S_{\mathfrak{F}_{12}}(\omega \circ \varepsilon_1) - S_{\mathcal{A}_1 \otimes \mathfrak{F}_2}(\omega)) + (S_{\mathfrak{F}_S \vee G_\tau}(\omega \circ \varepsilon_\tau) - S_{\mathfrak{F}_S}(\omega)). \end{aligned} \quad (7.155)$$

Since the conditional expectation ε_τ does not preserve the trace unless the group is Abelian, we cannot convert the first two terms into a RE using (7.105). However, we can use the fact that $\omega \circ \varepsilon_\tau$ is a product state on $\mathfrak{F}_S \otimes G_\tau$. In fact, this state is equal to $\omega_S \otimes \varphi$, where

⁴⁶This is non-longer true in the lattice QFT.

$\omega_S := \omega|_{\mathfrak{F}_S}$ and φ is the state on G_τ defined by $\varphi(\tau_g) := \delta_{g,1}$. Then, we write

$$S_{\mathfrak{F}_S \vee G_\tau}(\omega) - S_{\mathfrak{F}_S \vee G_\tau}(\omega \circ \varepsilon_\tau) = -S_{\mathfrak{F}_S \vee G_\tau}(\omega|\omega \circ \varepsilon_\tau) + \Delta\langle K_\varphi \rangle, \quad (7.156)$$

where $\Delta\langle K_\varphi \rangle := -\omega(\log \rho_\varphi^{G_\tau}) + \varphi(\log \rho_\varphi^{G_\tau})$. Replacing this last equation into (7.155)

$$\begin{aligned} & S_{\mathfrak{F}_{12}}(\omega|\omega \circ \varepsilon_{12}) + S_{\mathfrak{F}_S \vee G_\tau}(\omega \circ \varepsilon_\tau) \\ &= \Delta\langle K_\varphi \rangle + (S_{\mathfrak{F}_{12}}(\omega \circ \varepsilon_1) - S_{\mathcal{A}_1 \otimes \mathfrak{F}_2}(\omega)) + (S_{\mathfrak{F}_S \vee G_\tau}(\omega \circ \varepsilon_\tau) - S_{\mathfrak{F}_S}(\omega)). \end{aligned} \quad (7.157)$$

Both two last terms within parenthesis on the r.h.s. are formed by differences of entropies between states that are invariant under the conditional expectations but computed in the algebra and its fix-point subalgebra. The last term within brackets gives the entropy of the state φ on the group algebra G_τ , since the state $\omega \circ \varepsilon_\tau$ is a tensor product state on $\mathfrak{F}_S \otimes G_\tau$. To compute it we note that the group algebra is a sum of full matrix algebras

$$G_\tau \simeq \bigoplus_{\sigma \in \hat{G}} M_{d_\sigma}(\mathbb{C}) = \mathfrak{A}_G, \quad (7.158)$$

which dimensions d_σ corresponding to the irreducible representations $\sigma \in \hat{G}$. The projectors onto the different blocks are the P_σ in (7.86), which have expectation values $\varphi(P_\sigma) = d_\sigma^2/|G|$. Then, the density matrix is block diagonal with elements $d_\sigma/|G|$ on the diagonal in each block. The entropy is then

$$S_{\mathfrak{F}_S \vee G_\tau}(\omega \circ \varepsilon_\tau) - S_{\mathfrak{F}_S}(\omega) = S_{G_\tau}(\varphi) = \log |G| - \sum_{\sigma \in \hat{G}} \frac{d_\sigma^2}{|G|} \log d_\sigma. \quad (7.159)$$

To evaluate the first parenthesis on the r.h.s. of (7.157), we must notice that $\mathcal{A}_1, G_\tau \subset \mathfrak{F}_1$ and they are respective commutants inside inside \mathfrak{F}_1 , i.e.

$$G_\tau = \mathcal{A}'_1 \cap \mathfrak{F}_1 \quad \text{and} \quad \mathcal{A}_1 = G'_\tau \cap \mathfrak{F}_1, \quad (7.160)$$

and its common center $\mathcal{Z}(\mathcal{A}_1) = \mathcal{Z}(G_\tau)$ is again formed by the algebra of projectors P_σ spanning the center of the group algebra. Then, diagonalizing these projectors, we have a representation $\mathcal{A}_1 \vee G_\tau \cong \bigoplus_{\sigma \in \hat{G}} M_{d_\sigma}(\mathbb{C}) \otimes M_{n_\sigma}(\mathbb{C})$, where the group acts as $D_\sigma(g)$ in each block in the first factor and $M_{n_\sigma}(\mathbb{C})$ represents matrix algebras of invariant operators of \mathcal{A}_1 . An invariant state like $\omega \circ \varepsilon_1$ has a density matrix

$$\rho_{\omega \circ \varepsilon_1}^{\mathfrak{F}_1} = \bigoplus_{\sigma \in \hat{G}} \left(q_\sigma \frac{\mathbf{1}_{d_\sigma}}{d_\sigma} \otimes \rho_\sigma \right), \quad (7.161)$$

where $q_\sigma := \omega(P_\sigma)$ are the frequencies with which each irreducible sector appears in the

algebra \mathfrak{F}_1 , and ρ_σ are density matrices in $M_{n_\sigma}(\mathbb{C})$. Then, we get

$$S_{\mathfrak{F}_{12}}(\omega \circ \varepsilon_1) - S_{\mathcal{A}_1 \otimes \mathfrak{F}_2}(\omega) = \sum_{\sigma \in \hat{G}} q_\sigma \log d_\sigma. \quad (7.162)$$

Moreover, taking into account that the vacuum state is invariant under global group symmetries, we have that the statistical operator corresponding to the state $\omega|_{G_\tau}$ must be

$$\rho_{\omega|_{G_\tau}} = \bigoplus_{\sigma \in \hat{G}} q_\sigma \frac{\mathbf{1}_{d_\sigma}}{d_\sigma}. \quad (7.163)$$

Then, we can write

$$\Delta \langle K_\varphi \rangle = - \sum_{\sigma \in \hat{G}} q_\sigma \log d_\sigma + \sum_{\sigma \in \hat{G}} \frac{d_\sigma^2}{|G|} \log d_\sigma. \quad (7.164)$$

Therefore, replacing (7.159), (7.162), and (7.164) into (7.157), we finally arrive to

$$S_{\mathfrak{F}_{12}}(\omega|\omega \circ \varepsilon_{12}) = \log |G| - S_{\mathfrak{F}_S \vee G_\tau}(\omega|\omega \circ \varepsilon_\tau). \quad (7.165)$$

Since this relation holds in any lattice discretization, it should also hold in the continuum limit. This is because the terms in the equation are all well-defined in such a limit. Finally, collecting all the results together, we arrive to

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) = \log |G| - S_{\mathfrak{F}_S \vee G_\tau}(\omega|\omega \circ \varepsilon_\tau). \quad (7.166)$$

If W_1 and W_2 become larger, the l.h.s. of (7.166) increases and the RE on the shell must decrease. This is why the RE on the algebra $\mathfrak{F}_S \vee G_\tau$ appears with a minus sign in (7.166).

Equation (7.166) again expresses an upper bound of $\log |G|$ to ΔI , but this is a finer upper bound since it is improved by the RE on the r.h.s. of (7.166). As in the case of the intertwiners, we can take any subalgebra $\mathcal{B}_{S_\tau} \subset \mathcal{F}_S \vee G_\tau$ to get a convenient upper bound

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) \leq \log |G| - S_{\mathcal{B}_{S_\tau}}(\omega|\omega \circ \varepsilon_\tau). \quad (7.167)$$

Any set of twists τ_g that close a representation of the group form a linear basis for some choice of the algebra G_τ . Then, we can restrict to this algebra.⁴⁷ In this case, recalling that in the twist algebra the states $\omega|_{G_\tau}$ and φ are represented by the statistical operators

$$\rho_{\omega|_{G_\tau}} := \bigoplus_{\sigma \in \hat{G}} q_\sigma \frac{\mathbf{1}_{d_\sigma}}{d_\sigma} \quad \text{and} \quad \rho_\varphi := \bigoplus_{\sigma \in \hat{G}} \frac{d_\sigma}{|G|} \mathbf{1}_{d_\sigma}, \quad (7.168)$$

⁴⁷Note that these are smeared twists, as spread as possible, to increase their expectation values, in contrast to the sharp twists we have used in the derivation above.

we get

$$I_{\mathfrak{F}}(W_1, W_2) - I_A(W_1, W_2) \leq - \sum_{\sigma \in \hat{G}} q_\sigma \log q_\sigma + \sum_{\sigma \in \hat{G}} q_\sigma \log(d_\sigma^2). \quad (7.169)$$

Equation (7.169) is the same as expression (7.129), which is positive and bounded from above by $\log |G|$. It is a function of the twist expectation values through (see (7.86))

$$q_\sigma = \langle P_\sigma \rangle = \frac{d_\sigma}{|G|} \sum_{g \in G} \chi_\sigma^*(g) \langle \tau_g \rangle. \quad (7.170)$$

We get $\log |G|$ on the r.h.s. of (7.167) for “sharp” twists satisfying $\langle \tau_g \rangle = \delta_{g,1}$. These expectation values imply the regular representation probabilities through the previous relation. In a realistic scenario, the smallest upper bound will be for the most spread out twists, where the expectation values of the twists are bigger and the RE on the twist algebra is larger. On the other side of the story, the upper bound goes to zero when $\langle \tau_g \rangle = 1$ for all τ_g . This is the case for the vacuum and the global group transformations, which satisfy $q_\sigma = \delta_{\sigma, \text{triv}}$. Finally, we notice that for Abelian groups (7.169) is just the entropy on the twist algebra since the second term vanishes. This is not the case of a non-Abelian group where the entropy in the twist algebra is $-\sum_{\sigma \in \hat{G}} q_\sigma \log q_\sigma + \sum_{\sigma \in \hat{G}} q_\sigma \log(d_\sigma)$ rather than (7.169). Hence, there is an additional contribution in (7.169). This is necessary to match the intertwiner RE in the special cases where the upper and the lower bounds coincide.

We want to remark that the expression (7.169) for an upper bound should remain valid for continuous groups as far as the group is compact and the statistics of the sectors give a finite result. In fact, it will turn out this expression is generally finite for Lie group symmetries in QFT.

7.2.4. Entropic certainty relation

We recall equation (7.165)

$$S_{\mathfrak{F}_{12}}(\omega|\omega \circ \varepsilon_{12}) + S_{\mathfrak{F}_S \vee G_\tau}(\omega|\omega \circ \varepsilon_\tau) = \log |G|. \quad (7.171)$$

For large intertwiner expectation values (small separation between the regions W_1 and W_2), the first relative entropy will approach $\log |G|$, implying the twist one goes to zero, while the opposite is true for large separation between the regions, where there are some twists with large expectation values. Equation (7.171) is an *entropic certainty relation*. We remark a close relation between the algebras involved in such a relation, that it could

be represented in the following diagram

$$\begin{array}{ccc}
 \mathfrak{F}_{12} & \xrightarrow{\varepsilon_{12}} & \mathcal{A}_{12} \\
 \downarrow \prime & & \downarrow \prime \\
 \mathfrak{F}_S & \xleftarrow{\varepsilon_\tau} & \mathfrak{F}_S \vee G_\tau.
 \end{array} \tag{7.172}$$

The horizontal arrows indicates the action of the conditional expectations, whereas the vertical arrows indicates commutants. In other words, the initial space of the conditional expectation ε_τ coincides with the commutant of the target space of ε_{12} and viceversa. We also have that the underlying state is pure in the global algebra \mathfrak{F} . Moreover, there is a strong relation between the number $|G|$ and the algebras \mathfrak{F}_{12} and \mathcal{A}_{12} . Given any inclusion of vN algebras and a conditional expectation

$$\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H}), \quad \varepsilon : \mathcal{M} \rightarrow \mathcal{N}, \tag{7.173}$$

it can be defined an algebraic index $[\mathcal{M} : \mathcal{N}]_\varepsilon \geq 1$ which measures, in some sense, the “size” of the portion of \mathcal{M} that is killed by the conditional expectation ε [43, 44, 72]. In our case, we have exactly $[\mathfrak{F}_{12} : \mathcal{A}_{12}]_{\varepsilon_{12}} = |G|$. Moreover, we believe that the structure (7.172) can be extended to more general scenarios as follows. Given the general inclusion of vN algebras and the conditional expectation (7.173), it should exists a “dual” conditional expectation $\varepsilon' : \mathcal{N}' \rightarrow \mathcal{M}'$, with the same index $[\mathcal{M} : \mathcal{N}]_\varepsilon = [\mathcal{N}' : \mathcal{M}']_{\varepsilon'}$, such that the following relation holds

$$S_{\mathcal{M}}(\omega|\omega \circ \varepsilon) + S_{\mathcal{N}}(\omega|\omega \circ \varepsilon') = \log [\mathcal{M} : \mathcal{N}]_\varepsilon, \tag{7.174}$$

for any pure state on $\mathcal{B}(\mathcal{H})$. We are not sure if this is true for any given conditional expectation $\varepsilon : \mathcal{M} \rightarrow \mathcal{N}$, but perhaps for the one that minimizes the index, which gives the lower possible value on the r.h.s. of (7.174). In the case of (7.172), the conditional expectations are the unique possible ones for such an inclusion of algebras. This conjecture is left for future work.

If in (7.171) we restrict to subalgebras \mathcal{B}_{12} and \mathcal{B}_{G_τ} , we obtain instead the *entropic uncertainty relation*

$$S_{\mathcal{B}_{12}}(\omega|\omega \circ \varepsilon_{12}) + S_{\mathcal{B}_{G_\tau}}(\omega|\omega \circ \varepsilon_\tau) \leq \log |G|. \tag{7.175}$$

In order to both relative entropies being non-zero, \mathcal{B}_{12} (resp. \mathcal{B}_{G_τ}) must contain, at least, some non-trivial subalgebra of intertwiners (resp. twists).

Similar entropic uncertainty relations occur for generalized measurements [156, 157]. Notice that the maximal RE for each term needs minimal uncertainty: expectation values of the twist operators or the intertwiners equal to maximal ones. In the case of mini-

mal uncertainty, each RE can achieve $\log |G|$. Therefore, minimal uncertainty cannot be achieved at the same time for intertwiners and twists. The non-trivial commutation relations between twists and intertwiners is what prevents the left-hand side of this inequality to reach $2 \log |G|$, while $\log |G|$ is the maximum that can be achieved for each of the two terms separately.

In the same way, if we have an impure global state that is invariant under the group transformations (i.e. a thermal state), we can purify it in a larger Hilbert space (GNS representation) and upon reduction we get

$$S_{\tilde{\mathfrak{F}}_{12}}(\omega|\omega \circ \varepsilon_{12}) + S_{\tilde{\mathfrak{F}}_S \vee G_\tau}(\omega|\omega \circ \varepsilon_\tau) \leq \log |G|. \quad (7.176)$$

Uncertainty relations may be derived for operator expectation values rather than entropies using the commutations relations between twists and intertwiners. For example, in the case of the bosonic subnet of the fermion net described above, we have just one twist and one intertwiner satisfying

$$\tau \mathcal{I}_{12} = -\mathcal{I}_{12} \tau. \quad (7.177)$$

The usual uncertainty relation for non-commuting operators gives

$$|\langle \tau \rangle|^2 + |\langle \mathcal{I}_{12} \rangle|^2 \leq 1. \quad (7.178)$$

Then, when the twist has maximal expectation value $|\langle \tau \rangle| = 1$, the expectation value of the intertwiner is zero, and viceversa.

It is important to emphasize, that relation (7.171) holds not only for double cones but also for any pair of strictly spacelike separated topological trivial regions. The better way to understand such a kind of regions is fixing a Cauchy surface $\Sigma \subset \mathbb{R}^d$ and taking regions $\mathcal{O}_j := D(\mathcal{C}_j)$ with spacelike regions $\mathcal{C}_j \subset \Sigma$. Then, (7.171) holds for regions \mathcal{C}_j which are topologically equivalent to a sphere or to the complement of a sphere.⁴⁸

In fact, DHR intertwiners are non-trivial objects for disconnected regions. Let us assume that the regions $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ are connected and strictly spacelike separated. The set of DHR intertwiners between \mathcal{O}_1 and \mathcal{O}_2 would be always the same regardless of the topology of \mathcal{O}_1 and \mathcal{O}_2 whenever both \mathcal{O}_1 and \mathcal{O}_2 remain connected. However, for other topologies for the connected regions \mathcal{O}_1 and \mathcal{O}_2 , a new different type of intertwiners may arise. However, these intertwiners do not correspond to sectors localized in double cones [139]. To summarize, relation (7.171) holds always for topological trivial regions, and it also holds just for connected regions whenever the theory has no other localized sectors rather than the DHR ones. Of course the same happens for relations (7.133) and (7.166).

⁴⁸In algebraic topology, all the homotopy groups of the topological trivial region \mathcal{C}_j are trivial, i.e. $\pi_n(\mathcal{C}_j) = 0$ for all $n = 1, \dots, d-1$.

In section 7.2.6, we study how the order parameter ΔI behaves for regions having different non-trivial topologies.

7.2.5. Lie groups

When the symmetry group is not finite, $\Delta I(W_1, W_2)$ will be divergent in the limit when the regions touch each other. The interest lies in understanding how this quantity depends on the distance ϵ between the two regions W_1 and W_2 , when $\epsilon \rightarrow 0^+$. This is important, for example, when we want to compute the universal regularized EE explained in section 4.2.3. With that idea in mind and for simplicity, we consider regions $W_1 := D(C_R)$ and $W_2 := D(C'_{R+\epsilon})$ ($\epsilon > 0$). Let us first analyze the case of a group $G := U(1)$. We have a continuum of twists

$$\tau_k := e^{ikQ_1}, \quad k \in [-\pi, \pi), \quad (7.179)$$

where Q_1 is the selfadjoint generator of the twist algebra crossing the region W_1 . Since the group G is compact, the group representation (7.179) decomposes into irreducible representations

$$\mathcal{H} = \bigoplus_{q \in \mathbb{Z}} \mathcal{H}_q, \quad \tau_k = \sum_{q \in \mathbb{Z}} e^{ikq} P_q, \quad (7.180)$$

where P_q is the projector in the subspace \mathcal{H}_q .

In general, computing the exact operators Q_1 and the expectation values of the twists on a specific theory will be a problem depending on the dynamics. However, we are interested in the limit $\epsilon \rightarrow 0^+$, and we will argue that the leading divergent term is universal. We know that inside the double cone W_1 , the operator Q_1 may be expressed as

$$Q_1 \sim \int d^d x J^0(x) \alpha(x), \quad (7.181)$$

where $J^0(x)$ is the charge density and $\alpha(x)$ is a convenient smearing function. This function integrates to 1 in time, it is spatially constant inside W_1 and vanishes along W_2 . On the shell $S := (W_1 \vee W_2)'$ the operator content and the smearing changes, such as to give τ_k the desired group properties.

For small ϵ , the leading term of the total charge fluctuation inside the ball will come from short distance charge fluctuations distributed all along the entanglement surface, with a particle-antiparticle on each side the corridor separating the two regions, see figure 7.1. We can then picture the fluctuations of the total charge contributing to Q_1 as given by a large sum of independent random variables $Q_{1,n}$ ($n = 1, \dots, N$ and $N \gg 1$), since short distance fluctuations that are separated by a macroscopic distance along the surface of the sphere will not see each other.⁴⁹ Then, due to the central limit theorem, the total charge

⁴⁹We will come back to this point in section 7.3.4, where we elaborate a little more about general

distribution $Q_1 := \sum_{n=1}^N Q_{1,n}$ can be very well approximated by a Gaussian distribution

$$p_q := \langle P_q \rangle \simeq \frac{c}{\sqrt{2\pi\langle Q_1^2 \rangle}} e^{-\frac{q^2}{2\langle Q_1^2 \rangle}}, \quad c > 0, \quad (7.182)$$

where $\langle Q_1^2 \rangle = \sum_{n=1}^N \langle Q_{1,n}^2 \rangle \gg 1$ for small ϵ . Therefore, replacing these probabilities into (7.180), we get

$$\langle \tau_k \rangle = \sum_{q \in \mathbb{Z}} p_q e^{iqk} \simeq e^{-\frac{1}{2}k^2 \langle Q_1^2 \rangle}, \quad (7.183)$$

where we have used that $\langle Q_1^2 \rangle \gg 1$ in order to approximate the above expression using a continuous Fourier transform. This is why the result is not periodic in k , but it will hold very approximately in the limit we are studying.

An upper bound for ΔI is then easily computed from (7.169), to be the entropy of this distribution⁵⁰

$$-\sum_{q \in \mathbb{Z}} p_q \log(p_q) = 1/2 \log \langle Q_1^2 \rangle + \text{constant}. \quad (7.186)$$

Notice that even if the twist algebra has a continuum of operators, the upper bound is well-defined because it is the entropy of a classical discrete set of charges, or equivalently, because the group is compact. We expect that the difference of MIs is divergent in the non-compact case. There should be no problem with the MI on \mathcal{A} , but however, the one on \mathfrak{F} is the one not well-defined in this case. We think that the problem is that \mathfrak{F} contains too many sectors that would make fail the splitting property that guarantees we can take the algebra of two regions as a tensor product. This splitting property is related to the finiteness of a nuclearity index [2], which in turn is related to the partition function. Similar observations have been made recently using other arguments [158].

The best upper bound corresponds to the lowest $\langle Q_1^2 \rangle$. This corresponds to the most spread out twists. As the smearing function on the shell becomes wider, the probability of charge fluctuations on each side of the shell decreases, and the charge fluctuations inside the smearing region are averaged to zero. We give a more direct calculation of $\langle Q_1^2 \rangle$ in section 7.3.4 below. Here we notice that the result must be proportional to the area since bulk virtual fluctuations of the charge are suppressed because they will appear with both

properties of twists expectation values.

⁵⁰More precisely, the central limit theorem asserts that

$$\sum_{q \leq q_0} p_q \simeq \int_{-\infty}^{q_0} dq \frac{1}{\sqrt{2\pi\langle Q_1^2 \rangle}} e^{-\frac{q^2}{2\langle Q_1^2 \rangle}}, \quad \langle Q_1^2 \rangle \gg 1, \quad (7.184)$$

and moreover, the characteristic function (7.183) converges pointwise to

$$\sum_{q \in \mathbb{Z}} p_q e^{iqk} \simeq \int_{-\infty}^{q_0} dq \frac{1}{\sqrt{2\pi\langle Q_1^2 \rangle}} e^{-\frac{q^2}{2\langle Q_1^2 \rangle}} e^{ikq} = e^{-\frac{1}{2}k^2 \langle Q_1^2 \rangle}, \quad \langle Q_1^2 \rangle \gg 1, \quad k \ll 1. \quad (7.185)$$

signs and the total charge average zero. For a current that is conformal in the UV, the area A must be compensated by powers of the cutoff, giving $\langle Q_1^2 \rangle \sim A/\epsilon^{d-2}$. Then, we have that

$$\Delta I \leq \frac{1}{2} \log \frac{A}{\epsilon^{d-2}} + \text{constant} \sim \frac{(d-2)}{2} \log \frac{R}{\epsilon} + \text{constant}. \quad (7.187)$$

A lower bound can be given by using the intertwiners. There is one intertwiner \mathcal{I}_q for each number $q \in \mathbb{Z}$ representing the charge, which labels the irreducible representations of the group. This Abelian intertwiner algebra \mathcal{B}_{12} is C^* -isomorphic to the algebra of continuous periodic functions in the interval in \mathbb{R} , or equivalently, to the algebra of continuous functions in the unit circle S^1 .⁵¹ Each intertwiner is represented by the function

$$\mathcal{I}_q \mapsto f_q(k) := e^{ikq}, \quad k \in [-\pi, \pi). \quad (7.188)$$

In fact, given any state ψ on the Abelian algebra \mathcal{B}_{12} , (\mathcal{B}_{12}, ψ) corresponds to the classical probability space $(S^1, \mathcal{B}(S^1), p_\psi(k) dk)$ where the probability measure $p_\psi(k) dk$ is given by the probability density⁵²

$$p_\psi(k) := \frac{1}{2\pi} \sum_{q \in \mathbb{Z}} e^{ikq} \langle \mathcal{I}_q \rangle_\psi, \quad k \in [-\pi, \pi). \quad (7.189)$$

Then, this Abelian algebra corresponds to a continuous classical random variable on the space $[-\pi, \pi)$. Furthermore, the (quantum) relative entropy between two states ψ and ϕ on \mathcal{B}_{12} is equivalent to the classical relative entropy (Kullback–Leibler divergence) between the classical probabilities densities $p_\psi(k)$ and $p_\phi(k)$,

$$S_{\mathcal{B}_{12}}(\psi | \phi) = H_{KL}(p_\psi | p_\phi) := \int_{-\pi}^{\pi} p_\psi(k) \log \left(\frac{p_\psi(k)}{p_\phi(k)} \right). \quad (7.190)$$

In the present scenario, the state $\omega \circ \varepsilon_{12}$ is represented by the constant probability density $p_{\omega \circ \varepsilon_{12}}(k) := 1/(2\pi)$ since we have that $\omega \circ \varepsilon_{12}(\mathcal{I}_q) = 0$ for all $q \neq 0$. The probability density $p_\omega(k)$ corresponding to the state ω depends on the specific algebra of intertwiners we choose. We have to select it in order to maximize the relative entropy $S_{\mathcal{B}_{12}}(\omega | \omega \circ \varepsilon_{12})$. This can be achieved by concentrating the probability $p_\omega(k)$ around $k = 0$ as much as possible. This means that the probability of the different charges is as much flat as possible. In particular, to sense the probability distribution (7.182) of charge vacuum fluctuations, our intertwiners will have to be spread out on the surface of the sphere. Any smaller localization will lead to a less flat distribution of probabilities of charges. Heuristically, the intertwiner \mathcal{I}_q will then carry a state $\sqrt{p_{q_0}} |q_0\rangle_1 \otimes | -q_0\rangle_2$ to $\sqrt{p_{q_0}} |q_0 + q\rangle \otimes | -q_0 - q\rangle_2$.

⁵¹For any Abelian group G the intertwiners are labeled by its representations, and we can represent the Abelian algebra of the intertwiners with the algebra of functions on the group. This coincides with the algebra of the characters, which has the product law $\chi_{r_1}(g)\chi_{r_2}(g) = \chi_{r_1 \otimes r_2}(g)$.

⁵² $(S^1, \mathcal{B}(S^1))$ corresponds to the measurable space over S^1 generated by its Borel sets $\mathcal{B}(S^1)$.

The expectation value will be

$$\langle \mathcal{I}_q \rangle \simeq \sum_{q_0 \in \mathbb{Z}} \sqrt{p_{q_0} p_{q_0+q}} = e^{-\frac{q^2}{8\langle Q_1^2 \rangle}}. \quad (7.191)$$

According to (7.189), the classical probability $p_\omega(k)$ is therefore

$$p_\omega(k) \simeq \sqrt{\frac{2\langle Q_1^2 \rangle}{\pi}} e^{-2\langle Q_1^2 \rangle k^2}, \quad \langle Q_1^2 \rangle \gg 1. \quad (7.192)$$

and following (7.190) the relative entropy $S_{\mathcal{B}_{12}}(\omega | \omega \circ \varepsilon_{12})$ gives the same leading order calculation (7.186). We then get that the asymptotic behavior is

$$\Delta I \simeq \frac{1}{2} \log \frac{A}{\epsilon^{d-2}} \sim \frac{(d-2)}{2} \log \frac{R}{\epsilon}, \quad \epsilon \rightarrow 0^+. \quad (7.193)$$

This term should be attributed to the model \mathcal{A} as a contribution $-\frac{(d-2)}{2} \log \frac{R}{\epsilon}$ to the MI. This logarithmic term is “topological” in the sense that it appears in any dimension $d > 2$, and it does not depend on the curvature of the boundary as the usual logarithmic anomaly terms.

In $d = 2$ we have to replace $(R/\epsilon)^{(d-2)} \rightarrow \log(R/\epsilon)$, and the leading term is

$$\Delta I \simeq \frac{1}{2} \log(\log(R/\epsilon)), \quad \epsilon \rightarrow 0^+. \quad (7.194)$$

However, in $d = 2$ this is correct for two intervals that touch each other, while in the case of nearly complementary regions the shell consists of two intervals and the coefficient gets duplicated for massive fields, while it is still (7.194) for CFTs (see section 7.2.9).

For a non-Abelian compact Lie group G , we have different twist generators L_i , $i = 1 \dots \mathcal{G}$, where \mathcal{G} is the dimension of its Lie algebra. For each of these charges we expect to have a Gaussian probability of charges as in (7.182) for the same reasons as above. The group is non-commutative though. However, the typical expectation values of the charges are very large in the limit of small ϵ , and therefore we are in the regime of “large numbers” where the non-commutativity is not relevant. Then, the intertwiner version gives us a picture of \mathcal{G} independent charges with

$$\Delta I \simeq \frac{1}{2} (d-2) \mathcal{G} \log \frac{R}{\epsilon}, \quad \epsilon \rightarrow 0^+. \quad (7.195)$$

The twist version matches this expectation but there is a subtlety. A twist $e^{ik_i L_i}$ has appreciable expectation value only for small parameters k_i as in (7.183). This means only the neighborhood of the identity is probed in the group. Therefore we might expect to have effectively the case of \mathcal{G} Abelian generators. This is correct, but the conditional

expectation knows that these different directions in the Lie algebra can be connected by group transformations and cannot be considered independent. Hence, the vN entropy in the group algebra is in fact smaller than what is expected for the case of \mathcal{G} Abelian generators. However, the formula (7.169) contains an additional piece on top of the twist entropy in the non-Abelian case, and taking into account this contribution, the calculation with the twists matches the expectation (7.195) from the intertwiners.

Let us see how this work in a concrete example. Consider the case of $G := SO(3)$. According to the discussion above, for small ϵ , the expectation values of the twist τ_g are non-zero only for those corresponding to group elements near the identity element. In this situation, as in the Abelian case, it is useful to parametrize the twist operators with a 3-vector \vec{k} according to

$$\tau_{\vec{k}} = e^{i\vec{k}\cdot\vec{L}}, \quad (7.196)$$

where $\vec{L} := (L_1, L_2, L_3)$ are like angular momentum operators with commutation relations $[L_j, L_k] = i\epsilon_{jkl}L_l$. As argued above, the vacuum expectation values of such twist operators in the small ϵ limit is Gaussian, and has to be rotationally invariant

$$\langle \tau_{\vec{k}} \rangle \simeq e^{-\frac{1}{2}|\vec{k}|^2 \langle \vec{L}^2 \rangle}. \quad (7.197)$$

Then, it behaves as if they were the twist operators associated with three independent generators of the Abelian group $U(1)^3$. The computation using these expectation values is straightforward. First, we have that the irreducible representations of $SO(3)$ are labeled by a non-negative integer $l \in \mathbb{Z}_{\geq 0}$. The l -representation has dimension $d_l := (2l + 1)$ and its character χ_l is given by [159]

$$\chi_l(\theta) := \frac{\sin\left(\left(l + \frac{1}{2}\right)|\theta|\right)}{\sin\left(\frac{1}{2}|\theta|\right)}, \quad (7.198)$$

where θ is the angle of rotation from the identity. This coincides with $\theta \simeq |\vec{k}|$ for small θ . To compute the desired upper bound using equation (7.169), we need first to calculate the probabilities q_l attached to the l -representation. For that we use the Lie group continuum version of (7.170)

$$q_l := (2l + 1) \frac{1}{\pi} \int_0^\pi dk (1 - \cos(\theta)) \chi_l(\theta) e^{-\frac{1}{2}\theta^2 \langle \vec{L}^2 \rangle}, \quad (7.199)$$

where the finite sum was replaced by the integral over the full group $SO(3)$ using the normalized Haar measure (see [159]), and we are assuming $\langle \vec{L}^2 \rangle \gg 1$. Replacing (7.198) into (7.199), we can compute analytically the probabilities, which are given in terms of Erf

functions. At the end, replacing such probabilities into (7.169) we can check

$$I_{\mathcal{F}}(1, 2) - I_{\mathcal{O}}(1, 2) \leq - \sum_{l=0}^{\infty} q_l \log(q_l) + \sum_{l=0}^{\infty} q_l \log(d_l^2) \sim \frac{3}{2} \log \langle \bar{L}^2 \rangle + \text{constant}, \quad (7.200)$$

as we have claimed above.

We notice that each term in (7.200), for large $\langle \bar{L}^2 \rangle$ (small ϵ), reads

$$- \sum_{l=0}^{\infty} q_l \log(q_l) \sim \frac{1}{2} \log \langle \bar{L}^2 \rangle, \quad (7.201)$$

$$2 \sum_{l=0}^{\infty} q_l \log(d_l) \sim \log \langle \bar{L}^2 \rangle. \quad (7.202)$$

For an invariant state ω , the density matrix for the twist algebra G_τ decomposes according to the irreducible representations as

$$\rho_\omega := \bigoplus_{l \in \mathbb{Z}_{\geq 0}} q_l \cdot \frac{\mathbf{1}_{d_l}}{d_l}, \quad (7.203)$$

where $\mathbf{1}_{d_l}$ is the identity matrix in the full matrix algebra $M_{d_l}(\mathbb{C})$. The vN entropy on this algebra is then

$$S_{G_\tau}(\omega) = - \sum_{l=0}^{\infty} q_l \log(q_l) + \sum_{l=0}^{\infty} q_l \log(d_l). \quad (7.204)$$

Then, this entropy contributes only with a $\log \langle \bar{L}^2 \rangle$ and the missing $1/2 \log \langle \bar{L}^2 \rangle$ comes from the fact that the last term in (7.204) has a factor 2 in the correct formula (7.200). This is in contrast with the Abelian case, where (7.200) gives the entropy in the twist algebra.

7.2.6. Other topologies

The same type of ideas can be used to try to understand the difference of MIs between the models \mathfrak{F} and \mathcal{A} for regions with different topologies, such as the one shown in figure 7.3. We remember that when we derived formula (7.101) using the conditional expectation property (3.65) for the two double cones W_1 and W_2 , we used the relation

$$(\phi_1 \otimes \phi_2) \circ (\varepsilon_1 \otimes \varepsilon_2) = \omega_1 \otimes \omega_2, \quad (7.205)$$

where ω_j and ϕ_j are the restriction of the vacuum state to the algebras $\mathfrak{F}(W_j)$ and $\mathcal{A}(W_j)$ respectively as in (7.98) and (7.99). However, this is no longer true if any of the regions are formed by different connected components. For example, let us suppose that W_1 is replaced for region $\mathcal{O}_1 := W_{1,1} \vee W_{1,2}$ where $W_{1,k}$ are strictly separated double cones. Now,

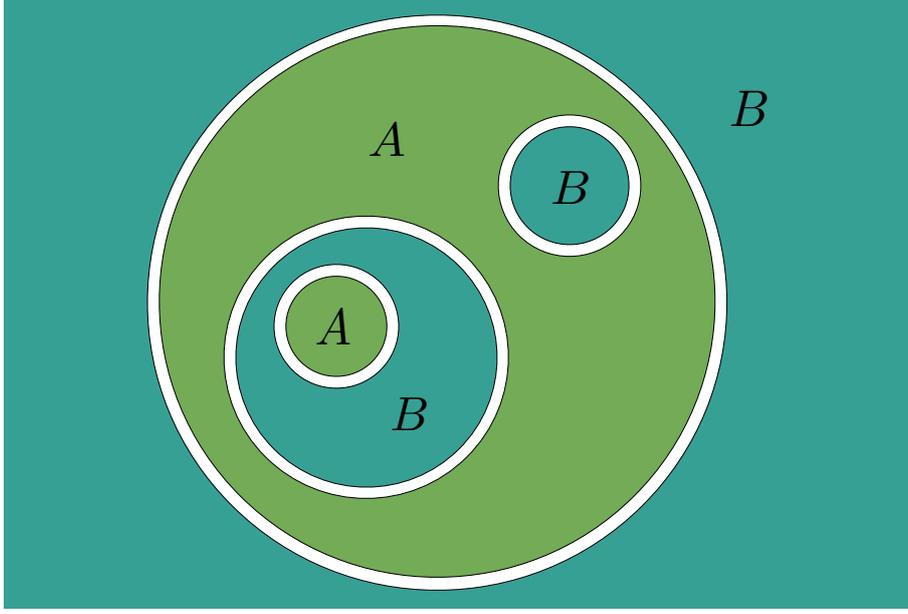


Figure 7.3: Two almost complementary regions A and B with non trivial topology. There is an independent set of intertwiners and twists for each connected component of the common boundary between A and B (white region). In this case, the number of connected components of the boundary is $n_{\partial} = 4$.

in this case, we may use of the conditional expectation $\varepsilon_1 : \mathfrak{F}(W_{1,1} \vee W_{1,2}) \rightarrow \mathcal{A}(W_{1,1} \vee W_{1,2})$

$$\varepsilon_1 (F_1 \hat{\otimes} F_2) = \varepsilon_{1,1} (F_1) \otimes \varepsilon_{1,1} (F_2) , \quad (7.206)$$

where $\varepsilon_{1,j} : \mathfrak{F}(W_{1,j}) \rightarrow \mathcal{A}(W_{1,j})$ are the usual conditional expectations of single double cones giving by the group average in each region $W_{1,j}$. Then, we have that $\phi_1 \circ \varepsilon_1 \neq \omega_1$ as states in $\mathfrak{F}(W_1)$, since $\omega_1(\mathcal{I}) \neq 0$ for intertwiners between the regions $W_{1,1}$ and $W_{1,2}$, whereas $\varepsilon_1(\mathcal{I}) = 0$ because it projects to the additive algebra $\mathcal{A}(\mathcal{O}_1) = \mathcal{A}(W_{1,1} \vee W_{1,2})$.

However, we can obtain a generalization of (7.101) complementing (3.65) with the property 8 of proposition 3.10. Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ be strictly spacelike separated regions, $\varepsilon_j : \mathfrak{F}(\mathcal{O}_j) \rightarrow \mathcal{A}(\mathcal{O}_j)$ be conditional expectations, $\varepsilon_{12} := \varepsilon_1 \otimes \varepsilon_2$ and $\omega_j, \phi_j, \omega_{12}$ and ϕ_{12} the restricted vacuum states as in (7.98) and (7.99). Then, we have that

$$\begin{aligned} I_{\mathfrak{F}}(\mathcal{O}_1, \mathcal{O}_2) - I_{\mathcal{A}}(\mathcal{O}_1, \mathcal{O}_2) &= S_{\mathfrak{F}}(\omega_{12} | \omega_1 \otimes \omega_2) - S_{\mathcal{A}}(\phi | \phi_1 \otimes \phi_2) \\ &= S_{\mathfrak{F}}(\omega_{12} | \omega_1 \otimes \omega_2) - S_{\mathfrak{F}}(\omega | (\omega_1 \otimes \omega_2) \circ \varepsilon_{12}) \\ &\quad + [S_{\mathfrak{F}}(\omega | (\omega_1 \otimes \omega_2) \circ \varepsilon_{12}) - S_{\mathcal{A}}(\phi | \phi_1 \otimes \phi_2)] , \end{aligned} \quad (7.207)$$

where for the term between brackets we have used the conditional expectation property (3.65) and for the second term we use $(\omega_1 \otimes \omega_2) \circ \varepsilon_{12} = (\omega_1 \circ \varepsilon_1) \otimes (\omega_2 \circ \varepsilon_2)$,

$$I_{\mathfrak{F}}(\mathcal{O}_1, \mathcal{O}_2) - I_{\mathcal{A}}(\mathcal{O}_1, \mathcal{O}_2) = S_{\mathfrak{F}}(\omega_{12} | \omega_1 \otimes \omega_2) - S_{\mathfrak{F}}(\omega | (\omega_1 \circ \varepsilon_1) \otimes (\omega_2 \circ \varepsilon_2)) + S_{\mathfrak{F}}(\omega | \omega \circ \varepsilon_{12}) . \quad (7.208)$$

Using property 8 of proposition 3.10 in the second term above, we finally arrive to

$$I_{\mathfrak{F}}(\mathcal{O}_1, \mathcal{O}_2) - I_A(\mathcal{O}_1, \mathcal{O}_2) = S_{\mathfrak{F}}(\omega_{12}|\omega \circ \varepsilon_{12}) - S_{\mathfrak{F}}(\omega_1|\omega_1 \circ \varepsilon_1) - S_{\mathfrak{F}}(\omega_2|\omega_2 \circ \varepsilon_2). \quad (7.209)$$

We remark that the first term on the l.h.s. of (7.209) is a RE on $\mathfrak{F}(\mathcal{O}_1 \vee \mathcal{O}_2)$, whereas the last two terms are REs in $\mathfrak{F}(\mathcal{O}_1)$ and $\mathfrak{F}(\mathcal{O}_2)$ respectively. When \mathcal{O}_1 and \mathcal{O}_2 are double cones, equation (7.209) coincides with (7.101) because the last terms two on the r.h.s of (7.209) vanish. However, (7.209) remains valid for any non-globally-invariant state ω .

Now we take a Cauchy surface $\Sigma \subset \mathbb{R}^d$ and two strictly spacelike separated regions $\mathcal{O}_A := D(A)$ and $\mathcal{O}_B := D(B)$, such that $A, B \subset \Sigma$ have n_A and n_B connected components respectively. Using (7.209) we get

$$\begin{aligned} I_{\mathfrak{F}}(\mathcal{O}_A, \mathcal{O}_B) - I_A(\mathcal{O}_A, \mathcal{O}_B) \\ = S_{\mathfrak{F}}(\omega_{AB}|\omega_{AB} \circ \otimes_i \varepsilon_{A_i} \otimes_j \varepsilon_{B_j}) - S_{\mathfrak{F}}(\omega_A|\omega_A \circ \otimes_i \varepsilon_{A_i}) - S_{\mathfrak{F}}(\omega_B|\omega_B \circ \otimes_j \varepsilon_{B_j}), \end{aligned} \quad (7.210)$$

where $i = 1, \dots, n_A$ and $j = 1, \dots, n_B$. In contrast to the single component case, the two last terms do not vanish for the vacuum state when $n_A, n_B > 1$. To obtain (7.210), we use that any connected region $\mathcal{O}_{A_i}, \mathcal{O}_{B_j}$ has a corresponding set of twist operators.

Let us try to understand the value of this MI difference. We are mainly interested in the limit case of two regions A and B that are nearly complementary to each other (see figure 7.3). Let us focus first on the first term on the right-hand side of (7.210). In general, there will be a complicated pattern of interference between the intertwiners crossing pairs of the $n_A + n_B$ regions, but under the current assumptions, this will be dominated by the intertwiners crossing between adjacent boundaries of A and B . In the particular limit we are considering here, each connected component of the boundary of A meets with a connected component of the boundary of B (see figure 7.3). Therefore, the number of connected components of the boundary of A and the ones of B agree. Let us call this number n_{∂} . Since each connected boundary divides the space Σ in two, the different boundaries form a tree under inclusion. This leads to the fact that the sets of intertwiners crossing each of these boundaries are algebraically independent to each other. They are also statistically independent because they are well localized in different boundaries. The total number of independent set of intertwiners is given by $n_A + n_B - 1$, because it is given by the total number of independent charge creating operator algebras, minus one to account for the neutrality of the operator. This coincides with the number of boundaries, $n_{\partial} = n_A + n_B - 1$, since in going from the interior to the exterior, each time we cross a boundary we can add a unique connected component of A or B . Then, the maximization of intertwiner entropy will indeed select the ones crossing each boundary with no free choices left, and we have

$$S_{\mathfrak{F}}(\omega_{AB}|\omega_{AB} \circ \otimes_i \varepsilon_{A_i} \otimes_j \varepsilon_{B_j}) = n_{\partial} \log |G|. \quad (7.211)$$

This last entropy has the form of a topological contribution.

There is a parallel twist version of this story. The difference between the entropies of the involved states in this relative entropy must come from the small region $C := (A \vee B)'$. This has exactly n_∂ connected components which are thin regions at the interfaces between A and B . Each of these connected surfaces divides the space in two and carries one set of independent twist operators that make the difference between the algebras of the two models in C . In the limit of small separation, independent sets of twists have all vanishing expectation value and are statistically independent. Hence again, each boundary contributes $\log |G|$.

Thinking in the mutual information difference in the generic case, the first term in (7.210) is bounded above by (7.211). From (7.210) we then have the general bound

$$0 \leq I_{\mathfrak{F}}(\mathcal{O}_A, \mathcal{O}_B) - I_A(\mathcal{O}_A, \mathcal{O}_B) \leq n_\partial \log |G|. \quad (7.212)$$

The second term on the right-hand side of (7.210) depends on the intertwiners crossing the different connected components of A , which are in the local algebra $\mathcal{A}'_\Sigma(A')$, but not in the algebra $\mathcal{A}_\Sigma(A)$. This term is in fact equal to the difference of generalized MIs⁵³

$$S_{\mathfrak{F}}(\omega_A | \omega_A \circ \otimes_i \varepsilon_{A_i}) = S_{\mathfrak{F}}(\omega_A | \omega_{A_1} \otimes \cdots \otimes \omega_{A_n}) - S_{\mathcal{A}}(\phi_A | \phi_{A_1} \otimes \cdots \otimes \phi_{A_n}). \quad (7.213)$$

This is in general difficult to compute but we can simplify this contribution if we focus in the case where the theory is gapped, and we are in the infrared regime with regions much larger than the gap scale. The expectation values of the intertwiners crossing components of A are exponentially small in the regime of large mass because the typical distances between components are large compared to the mass scale. In consequence, this contribution vanishes in this approximation. Equivalently, the same holds for the third term in (7.210), which concerns the multicomponent region B . Then, in this limit and for ϵ smaller than the gap scale, the bound is saturated and we obtain

$$I_{\mathfrak{F}}(\mathcal{O}_A, \mathcal{O}_B) - I_A(\mathcal{O}_A, \mathcal{O}_B) = n_\partial \log |G|. \quad (7.214)$$

7.2.7. Excitations

We want to investigate how the entropy on the algebra of a double cone W changes when we insert a well localized charged excitation. Evidently, in the case that the charged excitation corresponds to a sector of dimension $d_\sigma = 1$ (such as any excitation of an Abelian group symmetry), there will not be any change in the entropy because this is represented by an automorphism.

⁵³In finite dimensional algebras $S(\omega_A | \omega_{A_1} \otimes \cdots \otimes \omega_{A_n}) = S(\omega_{A_1}) + \cdots + S(\omega_{A_n}) - S(\omega_A)$.

Let us take the state ψ_σ^j on \mathfrak{F} given by the vector

$$|\psi_\sigma^j\rangle := \sqrt{d_\sigma} V_\sigma^{j\dagger} |0\rangle, \quad V_\sigma^j \in \mathfrak{F}(W), \quad (7.215)$$

corresponding to the irreducible representation $\sigma \in \hat{G}$ and already introduced in (7.70). For any element $A \in \mathcal{A}$, we have that

$$\langle \psi_\sigma^j | A | \psi_\sigma^j \rangle = \langle 0 | \rho_\sigma(A) | 0 \rangle = \omega \circ \rho_\sigma(A), \quad (7.216)$$

where

$$\rho_\sigma(A) := \sum_{j=1}^{d_\sigma} V_\sigma^j A V_\sigma^{j\dagger} \quad (7.217)$$

is the corresponding localized endomorphism. Notice that ψ_σ^j is pure in \mathfrak{F} but not in \mathcal{A} .

To measure the entropy of this impurity we can compute

$$S_{\mathfrak{F}_W}(\psi_\sigma^j | \psi_\sigma^j \circ \varepsilon) = S_{\mathfrak{F}_W}(\psi_\sigma^j | \omega \circ \rho \circ \varepsilon). \quad (7.218)$$

To get an upper bound to this quantity we can use 4 in proposition 3.10, and we get

$$S_{\mathfrak{F}_W}(\psi_\sigma^j | \psi_\sigma^j \circ \varepsilon) = S_{\mathfrak{F}_W}(\psi_\sigma^j \circ \alpha_g | \psi_\sigma^j \circ \varepsilon \circ \alpha_g) = S_{\mathfrak{F}_W}(\psi_\sigma^j \circ \alpha_g | \psi_\sigma \circ \varepsilon), \quad (7.219)$$

where α_g is the group automorphism (7.58). In the last equality we used that $\varepsilon = \varepsilon \circ \alpha_g$. Then, we can average over $g \in G$ the first state to obtain the second one. Even better, it is possible to average only over $G_\sigma := \{g_{\sigma,k} : k = 1, \dots, d_\sigma\} \subset G$ group elements, which just change the basis elements in the representation σ , to get the second state. Using again the convexity of the RE (5 in proposition 3.10), we get the upper bound

$$S_{\mathfrak{F}_W}(\psi_\sigma^j | \psi_\sigma^j \circ \varepsilon) = \frac{1}{d_\sigma} \sum_{g_\sigma \in G_\sigma} S_{\mathfrak{F}_W}(\psi_\sigma^j \circ \alpha_{g_\sigma} | \psi_\sigma \circ \varepsilon) \quad (7.220)$$

$$\leq S_{\mathfrak{F}_W} \left(\frac{1}{d_\sigma} \sum_{g_\sigma \in G_\sigma} \psi_\sigma^j \circ \alpha_{g_\sigma} | \psi_\sigma \circ \varepsilon \right) + \log(d_\sigma) = \log(d_\sigma). \quad (7.221)$$

The vectors $|\psi_\sigma^j\rangle$ are orthonormalized because they belong to different superselection sectors, i.e. $\langle \psi_\sigma^j | \psi_\sigma^k \rangle = \delta_{\sigma\sigma'} \delta_{jk}$. As long as we make the excitation support smaller, or the size of the double cone bigger, the reduced states for each $g_\sigma \in G_\sigma$ in the sum (7.221) become disjoint. This is the condition for the bound to become saturated

$$S_{\mathfrak{F}_W}(\psi_\sigma^j | \psi_\sigma^j \circ \varepsilon) = \log d_\sigma. \quad (7.222)$$

A completely localized state $\tilde{\psi}_\sigma^j$ on \mathfrak{F} is produced by the operators V_σ^j acting over the

vacuum vector, i.e.

$$|\tilde{\psi}_\sigma^j\rangle := V_\sigma^j|0\rangle, \quad V_\sigma^j \in \mathfrak{F}(W). \quad (7.223)$$

Contrary to (7.215), there is no factor $\sqrt{d_r}$ since $\langle \tilde{\psi}_\sigma^j | \tilde{\psi}_\sigma^k \rangle = \delta_{jk}$ because of $V_\sigma^{j\dagger} V_\sigma^k = \delta_{jk}$. The vector (7.223) corresponds to an excitation in the conjugate representation $\bar{\sigma}$. The state $\tilde{\psi}_\sigma^j$ on \mathcal{A} is equivalent to the state

$$\tilde{\psi}_\sigma := \frac{1}{d_\sigma} \sum_{\sigma=1}^{d_\sigma} \tilde{\psi}_\sigma^j. \quad (7.224)$$

In the same way as above, we have an upper bound $\log d_\sigma$ for the RE. The lower bound can be obtained by reducing to the finite dimensional subalgebra (7.119), generated by the operators $\{V_\sigma^j V_\sigma^{k\dagger} : j, k = 1, \dots, d_\sigma\}$. We have for the two states on this subalgebra

$$\langle \tilde{\psi}_\sigma^j | V_\sigma^k V_\sigma^{l\dagger} | \tilde{\psi}_\sigma^j \rangle = \delta_{jk} \delta_{jl}, \quad (7.225)$$

$$\frac{1}{d_\sigma} \sum_{j=1}^{d_\sigma} \langle \tilde{\psi}_\sigma^j | V_\sigma^k V_\sigma^{l\dagger} | \tilde{\psi}_\sigma^j \rangle = \frac{1}{d_\sigma} \delta_{kl}. \quad (7.226)$$

The RE on this subalgebra is $\log(d_r)$. Then, we have

$$S_{\mathfrak{F}W}(\tilde{\psi}_\sigma^j | \tilde{\psi}_\sigma^j \circ \varepsilon) = \log d_\sigma. \quad (7.227)$$

Instead of using this relative entropy we can try to compute the change in the mutual information for two touching double cones W_1 and W_2 . The excitation does not change correlations of operators in \mathfrak{F} outside the support of V_σ^j . Then, if this support is small and well-inside of W_1 , we expect that

$$I_{\mathfrak{F}}(W_1, W_2; \tilde{\psi}_\sigma^j) - I_{\mathfrak{F}}(W_1, W_2; \omega) \simeq 0. \quad (7.228)$$

To compute the change in the model \mathcal{A} we use the formula, which gives (7.209)

$$I_{\mathfrak{F}}(W_1, W_2; \tilde{\psi}_\sigma^j) - I_{\mathcal{A}}(W_1, W_2; \tilde{\psi}_\sigma^j) = S_{\mathfrak{F}_{12}}(\tilde{\psi}_\sigma^j | \tilde{\psi}_\sigma^j \circ \varepsilon_{12}) - S_{\mathfrak{F}_1}(\tilde{\psi}_\sigma^j | \tilde{\psi}_\sigma^j \circ \varepsilon_1) - S_{\mathfrak{F}_2}(\tilde{\psi}_\sigma^j | \tilde{\psi}_\sigma^j \circ \varepsilon_2). \quad (7.229)$$

The last term is zero because the two states are equal in W_2 . The second term is $\log(d_\sigma)$ as we have seen above. The first term is upper bounded by $\log |G| + \log d_\sigma$ since it is the minimal number of transformed states $\tilde{\psi}_\sigma^j$ we need to mix to get $\tilde{\psi}_\sigma^j \circ \varepsilon_{12}$. But it is also lower bounded by the same number since we can use a subalgebra formed by the one in (7.225-7.226) plus some intertwiner algebra near the touching boundary of W_1 and W_2 . Expectation values for these algebras are uncorrelated and the effect of the conditional

expectation can also be decoupled. Then, we conclude

$$I_{\mathfrak{F}}(W_1, W_2; \tilde{\psi}_\sigma^j) - I_{\mathcal{A}}(W_1, W_2; \tilde{\psi}_\sigma^j) = \log |G|. \quad (7.230)$$

Thus, the MI on \mathcal{A} does not change with respect to the vacuum, as happens on \mathfrak{F} . The excitation is impure in the model \mathcal{A} , but its impurity is due to a transformation of the vacuum well-inside the region W_1 , that it does not modify correlations with $\mathcal{A}(W_2)$. Therefore, it will not change the MI.

If we instead create the state $\varphi_{\sigma\bar{\sigma}}^j$, which represents a particle in W_1 and an antiparticle in W_2

$$|\varphi_{\sigma\bar{\sigma}}^j\rangle := V_{\sigma,1}^j V_{\bar{\sigma},2}^j |0\rangle, \quad V_{\sigma,1}^j \in \mathfrak{F}(W_1), V_{\bar{\sigma},2}^j \in \mathfrak{F}(W_2), \quad (7.231)$$

the MI on \mathfrak{F} will not change with respect to the vacuum one, because of the same reasons as above. For the model \mathcal{A} we can use again (7.229). Now, the last two REs are $\log d_\sigma$. The first one is again upper bounded and lower bounded by $\log |G| + \log d_\sigma$. Then, we conclude that

$$I_{\mathfrak{F}}(W_1, W_2; \varphi_{\sigma\bar{\sigma}}^j) - I_{\mathcal{A}}(W_1, W_2; \varphi_{\sigma\bar{\sigma}}^j) = \log |G| - \log d_\sigma, \quad (7.232)$$

$$I_{\mathcal{A}}(W_1, W_2; \varphi_{\sigma\bar{\sigma}}^j) - I_{\mathcal{A}}(W_1, W_2; \omega) = \log d_\sigma. \quad (7.233)$$

Remarkably this does not depend on how far separated are the excitations. We would have expected $2 \log d_\sigma$ for the MI of a maximally entangled state on a Hilbert space of dimension d_σ . But here, the effect is rather the MI of classical random variables with perfect correlation and maximal uncertainty (the effective state is analogous to (7.224) with $\tilde{\psi}_\sigma^j \rightarrow \varphi_{\sigma\bar{\sigma}}^j$), which gives half this number, and it is not produced by entanglement.

For the pure state $\varphi_{\sigma\bar{\sigma}}$ on \mathfrak{F} given by

$$|\varphi_{\sigma\bar{\sigma}}\rangle := \frac{1}{\sqrt{d_\sigma}} \sum_{j=1}^{d_\sigma} V_{\sigma,1}^j V_{\bar{\sigma},2}^j |0\rangle, \quad V_{\sigma,1}^j \in \mathfrak{F}(W_1), V_{\bar{\sigma},2}^j \in \mathfrak{F}(W_2), \quad (7.234)$$

we get, along the same lines,

$$I_{\mathfrak{F}}(W_1, W_2; \varphi_{\sigma\bar{\sigma}}) - I_{\mathcal{A}}(W_1, W_2; \varphi_{\sigma\bar{\sigma}}) = \log |G|. \quad (7.235)$$

This is because the state (7.234) is invariant under group automorphisms and hence, we have $S_{\mathfrak{F}_{12}}(\varphi_{\sigma\bar{\sigma}}|\varphi_{\sigma\bar{\sigma}} \circ \varepsilon_{12}) = \log |G|$ as for the vacuum state, and the two last terms in (7.229) vanish. Then, we expect

$$I_{\mathcal{A}}(W_1, W_2; \varphi_{\sigma\bar{\sigma}}) - I_{\mathcal{A}}(W_1, W_2; \omega) = I_{\mathfrak{F}}(W_1, W_2; \varphi_{\sigma\bar{\sigma}}) - I_{\mathfrak{F}}(W_1, W_2; \omega) = 2 \log d_\sigma, \quad (7.236)$$

as it corresponds to a pure state in both models.

Several results about the entropy of charged states, analogous to the ones studied in this section, have been previously appeared in the literature for specific models. See, for example, [55, 160–165].

7.2.8. Spontaneous symmetry breaking

If the observable algebra \mathcal{A} satisfies all the structural assumptions given in assumption 7.5, we can uniquely reconstruct the field algebra \mathfrak{F} and the compact symmetry group G , such that \mathcal{A} is made of the G -invariant elements of \mathfrak{F} . Moreover, the symmetry group acts unitarily in \mathcal{H} leaving the vacuum vector invariant. Then, there are no broken symmetries. In order to leave the possibility of the appearance of spontaneous symmetry breaking, we must relax some of the assumptions listed there. The manner this phenomenon is described in the algebraic setting is to impose that local algebras attached to double cones are no longer maximal in the vacuum sector, i.e. they do not satisfy the duality condition 3 of assumption 7.5. The local operators which are then missing are, roughly speaking, fields that transform non-trivially under the symmetry group but do not leave out of the vacuum sector [166–168].

A symmetry is said to be spontaneously broken if there is no unitary operator implementing it in the vacuum representation of \mathfrak{F} , or what amounts to the same thing, if the vacuum state is not stable under the action of such a symmetry.

Despite, the local algebras attached to double cones do not satisfy the duality condition, we assume they satisfy essential duality (see section 2.2.5). Then, we can define the dual net $\mathcal{O} \in \mathcal{K} \rightarrow \mathcal{A}^d(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0)$, acting over the same Hilbert space \mathcal{H}_0 where \mathcal{A} acts, as

- $\mathcal{A}^d(W) := \mathcal{A}^d(W')'$ for all double cones W ,
- and imposing additivity (condition 2 in assumption 7.5) for more general causally complete regions.

Then, we have that $\mathcal{A} \subset \mathcal{A}^d$ and, it can be shown, that the dual net satisfies all the axioms of definitions 2.56 and 2.58. In fact, the unitary representation of the Poincaré group and the vacuum vector of \mathcal{A} and \mathcal{A}^d are the same.

Any representation of \mathcal{A} fulfilling (7.27) extends canonically to representations of \mathcal{A}^d of the same type on the same Hilbert space [169]. Then, as we have explained in section 7.1.2, we can associate to \mathcal{A}^d a unique compact symmetry group H and a complete field system \mathfrak{F} on the Hilbert space \mathcal{H} . Then, \mathcal{A}^d is made of the H -invariant elements of \mathfrak{F} . We have then the inclusion of nets

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{A}^d(\mathcal{O}) \subset \mathfrak{F}(\mathcal{O}) . \quad (7.237)$$

Then, we can define the full symmetry group G as the group of all automorphisms of \mathfrak{F} leaving \mathcal{A} pointwise fixed. Of course, we have that $H \subset G$. Contrary to H , G may be non-compact. An element of G lies actually in H if and only if it corresponds to a non-spontaneously broken symmetry (i.e. it is unitarily implementable) if and only if it leaves \mathcal{A}^d pointwise fixed. Similarly, the elements of G that do not belong to H are the *spontaneously broken symmetries*. It can be shown that the elements of G leave all the local field algebras $\mathfrak{F}(\mathcal{O})$ globally stable, i.e.

$$\alpha(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O}), \quad \forall \alpha \in G, \forall \mathcal{O} \in \mathcal{K}, \quad (7.238)$$

and H is the stabilizer in G of the vacuum state on \mathfrak{F} .

Now, let us assume that G commutes with the Poincaré unitaries in \mathfrak{F} and leaves \mathcal{A}^d globally fixed.⁵⁴ Then, G leaves all the local algebras $\mathcal{A}^d(\mathcal{O})$ globally stable. We also have that H is a normal subgroup of G and G is an extension of H by G/H . The action of an element $g \in G$ takes the vacuum state, induced by the vector $|0\rangle \in \mathcal{H}_0$, to the state ω_g . Such a state coincides with the vacuum state over \mathcal{A} , but they may be different states over \mathcal{A}^d . Moreover, such a state is Poincaré-invariant in \mathfrak{F} . The set $\mathcal{V} := \{\omega_g : g \in G\}$ is the set of *degenerate vacua* and it is in one-to-one correspondence with G/H .

For simplicity, we assume that the group G is compact. Then, we can define two different conditional expectations

$$\varepsilon : \mathfrak{F} \rightarrow \mathcal{A} \quad \text{and} \quad \tilde{\varepsilon} : \mathfrak{F} \rightarrow \mathcal{A}^d, \quad (7.239)$$

given by the group averages on G and H respectively.

When a symmetry is broken it is not longer true that the RE for the vacuum on the two models is zero for a double cone W . This is because the vacuum expectation values do not generally vanish for the charged operators. In fact, the RE

$$S_{\mathfrak{F}_W}(\omega_g | \omega_g \circ \varepsilon), \quad (7.240)$$

is an interesting quantity to compute in this case and serves as an entropic order parameter for symmetry breaking. Using the invariance of the RE under automorphisms and the invariance of the conditional expectation ε under G , it can be shown that the above RE does not depend on the degenerate vacua chosen ω_g . Moreover, $S_{\mathfrak{F}_W}(\omega_g | \omega_{g'} \circ \varepsilon)$ is independent of $g, g' \in G$. From the rest of this section, ω denotes any vacua of \mathcal{V} .⁵⁵ Using (3.65) for $\tilde{\varepsilon} : \mathfrak{F}(W) \rightarrow \mathcal{A}^d(W)$ and the states ω and $\omega \circ \varepsilon$, we get $S_{\mathfrak{F}_W}(\omega | \omega \circ \varepsilon) = S_{\mathcal{A}^d_W}(\omega | \omega \circ \varepsilon)$.

Let us first discuss the case of a finite group G . As in section 7.2.3, the RE is upper

⁵⁴If it does not happen, we can consider the subgroup of G satisfying the above properties.

⁵⁵We do not consider a mixed state of vacua, since the model \mathfrak{F} does not satisfy clustering in such a state.

bounded because of the convexity property,

$$S_{\mathfrak{F}_W}(\omega|\omega \circ \varepsilon) \leq \log \left(\frac{|G|}{|H|} \right). \quad (7.241)$$

We have that $\omega = \omega \circ \tilde{\varepsilon}$, and to convert such a state into the second one on the l.h.s. of (7.241), we need to average only with $|G|/|H|$ independent elements of G/H . As a curiosity, we note that this is a RE for a single region, which is invariant under Poincaré transformations of the region. This rare luxury is possible precisely because of the existence of degenerate vacua.

Let $W := D(\mathcal{C}_R)$ with \mathcal{C}_R as in (7.4). The RE increases with the size R of the region W . We can take the entropy difference in any finite subalgebra stable under ε to get a lower bound. We expect that, for small R with respect to the scale of the symmetry breaking, the symmetry is effectively restored and the RE approaches to zero. In other words, there are no operators inside the region that are able to efficiently distinguish the two states. On the other hand, for regions larger than the symmetry breaking scale, we expect saturation of the bound (7.241). As an example, we can take a theory with a \mathbb{Z}_2 broken symmetry, where the order parameter is given by the non-zero expectation value of a scalar field $\phi(x)$. As an order parameter we can use a smeared mode $\phi_f := \int d^d x f(x)\phi(x)$ such that $\int d^d x f(x) = 1$. We have that $\langle \phi_f \rangle =: \mu$ in the state ω , and $\langle \phi_f \rangle = 0$ in the state $\omega \circ \varepsilon$. But the fluctuations of this mode for small support of the test function f will be much bigger than μ and of order of R^{-1} . Hence, we cannot efficiently distinguish the states in a small region.⁵⁶

To understand the behavior of the MI difference $\Delta I := I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2)$ we use formula (7.209)

$$\Delta I = S_{\mathfrak{F}_{12}}(\omega_{12}|\omega_{12} \circ \varepsilon_1 \otimes \varepsilon_2) - S_{\mathfrak{F}_1}(\omega_1|\omega_1 \circ \varepsilon_1) - S_{\mathfrak{F}_2}(\omega_2|\omega_2 \circ \varepsilon_2), \quad (7.242)$$

which is valid for general states. We are mainly interested in the case of nearly complementary regions $W_2 \rightarrow W'_1$. Then, let us consider $W_1 := D(\mathcal{C}_R)$ and $W_2 := D(\mathcal{C}'_{R+\epsilon})$ ($\epsilon > 0$). In this case, the last term, for an unbounded region W_2 , saturates to $\log(|G|/|H|)$. Using convexity we can show that

$$S_{\mathfrak{F}_{12}}(\omega_{12}|\omega_{12} \circ \varepsilon_1 \otimes \varepsilon_2) \leq 2 \log |G| - \log |H|. \quad (7.243)$$

In this case, to transform the state ω_{12} into $\omega_{12} \circ \varepsilon_1 \otimes \varepsilon_2$ we need to average with the elements of the group G in both regions independently, but we can dispense on average on $|H|$ elements in one of the regions, because the vacuum state is still globally invariant under the

⁵⁶To get a rough estimate of this behavior, we may assume Gaussian fluctuations. The RE, between a classical continuous variable with Gaussian distribution of width R^{-1} centered around the origin and another Gaussian distribution centered in μ , is $\sim (R\mu)^2$.

unbroken subgroup H .⁵⁷ In the present limit, we can argue that this term always saturates the bound (7.243). This is because we can use, as a lower bound, the RE on a subalgebra formed by charged operators localized far away from W_1 (e.g. the same subalgebra that one can use to show $S_{\mathfrak{F}_1}(\omega_2|\omega_2 \circ \varepsilon_2) = \log(|G|/|H|)$) and an intertwiner subalgebra around the common boundary of the regions W_1 and W_2 . Expectation values are independent for these subalgebras because the charged operators in the intertwiner subalgebra have large fluctuations, and we get the saturation of the bound (7.243) independently of the size R . Therefore, we expect

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) = \log |G| - S_{\mathfrak{F}_1}(\omega_1|\omega_1 \circ \varepsilon_1). \quad (7.244)$$

Hence, this is controlled by the same entropic order parameter discussed above. This approaches the result

$$\Delta I \rightarrow \begin{cases} \log |G|, & R\mu \ll 1, \\ \log |H|, & R\mu \gg 1, \end{cases} \quad \epsilon \ll R, \mu^{-1}, \quad (7.245)$$

where μ is the symmetry breaking scale, and ϵ is the separation between the regions W_1 and W_2 . The first case corresponds to the case when the broken symmetry is restored in the region W_1 , whereas the second case corresponds to the IR behavior where the symmetry is completely broken inside W_1 .

To understand the regime $\mu^{-1} \ll \epsilon, R$, when the separation between the regions is larger than the breaking scale, we have to split (7.242) into two contributions using the intermediate algebra \mathcal{A}^d

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}^d}(W_1, W_2) = S_{\mathfrak{F}_{12}}(\omega_{12}|\omega_{12} \circ \tilde{\varepsilon}_1 \otimes \tilde{\varepsilon}_2), \quad (7.246)$$

$$\begin{aligned} I_{\mathcal{A}^d}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) &= S_{\mathcal{A}_1^d}(\omega_{12}|\omega_{12} \circ \varepsilon_1 \otimes \varepsilon_2) \\ &\quad - S_{\mathcal{A}_1^d}(\omega_1|\omega_1 \circ \varepsilon_1) - S_{\mathcal{A}_2^d}(\omega_2|\omega_2 \circ \varepsilon_2). \end{aligned} \quad (7.247)$$

The first term is just the case when there are no broken symmetries, and it behaves as $I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}^d}(W_1, W_2) = \log |H|$ for $\epsilon \ll R$, and $I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}^d}(W_1, W_2) = 0$ for $\epsilon \gg R$. The second term is the case where all symmetries are broken. In this case, the group of these symmetries is the quotient group G/H .⁵⁸ For the same reasons as above, the first term in (7.247) is bounded above by $2 \log(|G|/|H|)$, whereas the second and third terms are each one bounded above by $\log(|G|/|H|)$. For $\mu^{-1} \ll R$, we have that both last

⁵⁷Despite the broken symmetries are not related to superselection sectors, there exist local implementations of the elements of G for any local algebra $\mathfrak{F}(\mathcal{O})$ for which the vacuum vector $|0\rangle$ is standard [168].

⁵⁸The conditional expectation $\varepsilon|_{\mathcal{A}^d}$ restricted to \mathcal{A}^d involves the average over $|G|/|H|$ independent elements of G because \mathcal{A}^d is H -invariant.

two terms saturate their respective bounds, pushing the first term in (7.247) to saturates since the r.h.s. of the same expression is positive due to monotonicity. Then, combining these results for (7.246) and (7.247), we obtain

$$\Delta I \rightarrow \begin{cases} \log |H|, & \epsilon \ll R, \\ 0, & \epsilon \gg R, \end{cases} \quad \mu^{-1} \ll \epsilon, R. \quad (7.248)$$

We remark, that in the case we want to interpret ΔI as a regularized entropy (see section 4.2.3), we must consider the regime (7.245) since we must take ϵ smaller than any other scale of the theory.

According to our analysis of the entropic certainty relation in section 7.2.4, we conclude that the following relation applies

$$S_{\mathfrak{F}_{12}}(\omega|\omega \circ \varepsilon_{12}) + S_{\mathfrak{F}_S \vee G_\tau}(\omega|\omega \circ \varepsilon_\tau) = 2 \log |G| - \log |H|. \quad (7.249)$$

The conditional expectation $\varepsilon_\tau : \mathfrak{F}_S \vee G_\tau \rightarrow \mathfrak{F}_S$ projects the local operators that implement (broken and unbroken) the symmetries in the region W_1 . It is the dual conditional expectation of ε_{12} for the inclusion of algebras $\mathcal{A}_{12} \subset \mathfrak{F}_{12}$, which has algebraic index $|G|^2/|H|$.⁵⁹

In this context, it is also interesting to consider the certainty relation for the SSB order parameter

$$S_{\mathfrak{F}_1}(\omega|\omega \circ \varepsilon_1) + S_{\mathfrak{F}_1' \vee G_\tau}(\omega|\omega \circ \varepsilon_{\tau_1}) = \log |G|, \quad (7.250)$$

which is represented in the diagram

$$\begin{array}{ccc} \mathfrak{F}(W_1) & \xrightarrow{\varepsilon_1} & \mathcal{A}(W_1) \\ \uparrow \prime & & \uparrow \prime \\ \mathfrak{F}(W_1') & \xleftarrow{\varepsilon_{\tau_1}} & \mathfrak{F}(W_1') \vee G_\tau. \end{array} \quad (7.251)$$

However, using the intermediate algebra \mathcal{A}^d , we have a different certainty relation

$$S_{\mathcal{A}_1^d}(\omega|\omega \circ \varepsilon_1) + S_{\mathcal{A}_1'^d}(\omega|\omega \circ \varepsilon_{\tau_1}) = \log |G| - \log |H|, \quad (7.252)$$

according to the diagram

$$\begin{array}{ccc} \mathcal{A}^d(W_1) & \xrightarrow{\varepsilon_1} & \mathcal{A}(W_1) \\ \uparrow \prime & & \uparrow \prime \\ \mathcal{A}^d(W_1)' = \mathcal{A}(W_1') & \xleftarrow{\varepsilon_{\tau_1}} & \mathcal{A}(W_1)' = \mathcal{A}^d(W_1'). \end{array} \quad (7.253)$$

⁵⁹This is because, in this case, to pass from \mathfrak{F}_{12} to \mathcal{A}_{12} we can use first the conditional expectation $\tilde{\varepsilon}_{12} : \mathfrak{F}_{12} \rightarrow \mathcal{A}_{12}^d$ which has index $|H|$ as in the unbroken case, and then the conditional expectation $\varepsilon_1 \otimes \varepsilon_2|_{\mathcal{A}_{12}^d} : \mathcal{A}_1^d \otimes \mathcal{A}_2^d \rightarrow \mathcal{A}_1 \otimes \mathcal{A}_2$ which has index $(|G|/|H|)^2$, because the average in each factor is independent and is given by the group G/H . The total index for the composition of the conditional expectations is the product of the indices and gives $|G|^2/|H|$.

We now argue why both certainty relations could be simultaneously valid despite we have that $S_{\mathfrak{F}_1}(\omega|\omega \circ \varepsilon_1) = S_{\mathcal{A}^d}(\omega|\omega \circ \varepsilon_1)$. This is because in (7.250) the commutants are taken inside the Hilbert space \mathcal{H} since all the field algebras involved are subalgebras $\mathcal{B}(\mathcal{H})$, whereas in (7.252) we used that $\mathcal{A} \subset \mathcal{A}^d \subset \mathcal{B}(\mathcal{H}_0)$ as long as we take commutants inside the vacuum Hilbert space \mathcal{H}_0 . For this reason, the commutant $\mathcal{A}^d(W'_1)$ of $\mathcal{A}(W_1)$ inside $\mathcal{B}(\mathcal{H}_0)$ does not include the DHR twists, but it includes the operators which implement the broken symmetries locally. On the other hand, the algebra $\mathfrak{F}(W'_1) \vee G_\tau$ include all the broken and unbroken twists. In this way, we have always $S_{\mathfrak{F}_1 \vee G_\tau}(\omega|\omega \circ \varepsilon_{\tau_1}) \geq \log |H|$ because there is a group subalgebra $H_\tau \subset G_\tau$ of $|H|$ independent twists of the unbroken part of G such that $S_{H_\tau}(\omega|\omega \circ \varepsilon_{\tau_1}) = \log |H|$ independently of the value of $R\mu$.

All the above discussion suits perfectly for a finite group G . In order to understand the case of SSB of a Lie group, we first study the simple model of a compactified free scalar.

7.2.8.1. Free compactified scalar

Let us take the global algebra \mathfrak{F} of the free massless scalar real field $\phi(x)$ for $d > 2$.⁶⁰ The way to construct this algebra is similar to what we have explained in chapter 5, with some small differences coming that, in this case, the mass is zero. The global algebra \mathfrak{F} is defined as a concrete C^* -algebra of operators acting on the vacuum Fock Hilbert space \mathcal{H}_0 . This Hilbert space is the symmetric tensor product of the Hilbert space of one-particle states of zero mass and zero spin. The local algebras are defined using the Weyl unitaries $e^{i\phi(f)}$, $f \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$, as

$$\mathfrak{F}(B) := \{e^{i\phi(f)} : \text{supp}(f) \subset B\}'' \subset \mathcal{B}(\mathcal{H}_0), \quad B \in \mathcal{K} \text{ bounded.} \quad (7.254)$$

For an unbounded region \mathcal{O} , we define $\mathfrak{F}(\mathcal{O})$ as the C^* -algebra generated by the unitaries $W(f)$ with bounded $\text{supp}(f) \subset \mathcal{O}$. This algebra satisfies duality for double cones (3 in assumption 7.5). Moreover, \mathfrak{F} has no superselection sectors and it also satisfies duality for any pair of double cones $W_1 \otimes W_2$.

We define the subalgebra $\mathcal{A} \subset \mathfrak{F}$ of the derivatives $\partial_\mu \phi(x)$ of the massless scalar field. This is the pointwise fixed subalgebra under the automorphisms

$$\alpha_s(\phi) := \phi + s, \quad s \in \mathbb{R}. \quad (7.255)$$

In this case, the group of such automorphisms is \mathbb{R} . They can be locally implemented for any standard region $\mathcal{O} \in \mathcal{K}$ [168]. Moreover, for a bounded region $B := D(\mathcal{C})$ with

⁶⁰For $d = 2$, this theory is not well-defined since it does not satisfies positivity. Then, we have to consider the subalgebra of the derivatives $\partial_\mu \phi$ of ϕ . But, this is unsatisfactory for the purpose of this subsection because we want to study the SSB $\phi \rightarrow \partial_\mu \phi$. This is consistent with the fact that SSB with continuous symmetry does not exists in $d = 2$. As we saw in chapter 6, in $d = 2$, the algebra of $\partial_\mu \phi$ is the invariant subalgebra of the free fermion field under the unbroken $U(1)$ -symmetry.

$\mathcal{C} \subset \Sigma_0 := \{x^0 = 0\} \subset \mathbb{R}^d$, we have that

$$\alpha_s(F) = e^{isQ_B} F e^{-isQ_B}, \quad F \in \mathfrak{F}(B), \quad (7.256)$$

$$Q_B := \int_{\mathbb{R}^{d-1}} J(\bar{x}) f_B(\bar{x}) d^{d-1}x, \quad (7.257)$$

where $J(\bar{x}) := \partial_0 \phi(x)|_{x^0=0}$ and $f_B \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ is any compactly supported function such that $f_B(\bar{x}) = 1$ for all $\bar{x} \in \mathcal{C}$.⁶¹ In term of the Weyl unitaries, (7.255) reads

$$\alpha_s(e^{i\phi(f)}) = e^{is \int_{\mathbb{R}^d} f(x) d^d x} e^{i\phi(f)}, \quad s \in \mathbb{R}. \quad (7.258)$$

Hence, the C^* -algebra \mathcal{A} is generated by the collection of all Weyl unitaries satisfying the subsidiary condition

$$\int_{\mathbb{R}^d} f(x) d^d x = 0. \quad (7.259)$$

Thereby, the local algebras of \mathcal{A} attached to double cones are defined as in (7.254) but restricting the smearing functions to the ones satisfying (7.259), whereas the local algebras for non-double cones are constructed from the double cone ones imposing additivity. The algebra \mathcal{A} acts irreducible in \mathcal{H}_0 . However, the local algebras attached to doubles cones are not maximal in the vacuum representation, i.e. they do not satisfy the duality condition 3 of assumption 7.5. As we discussed in the previous section, we shall consider the dual net $\mathcal{A}^d \supset \mathcal{A}$. In this example, it results that $\mathcal{A}^d \equiv \mathfrak{F}$. Then, we have that \mathcal{A} has no non-trivial superselection sectors and all the symmetries of are spontaneously broken.

In this case, the symmetry group is non-compact. In order to not deal with issues coming from non-compact groups,⁶² we consider the following intermediate algebra

$$\mathcal{A} \subsetneq \mathfrak{F}_\lambda \subsetneq \mathfrak{F}, \quad \lambda > 0, \quad (7.260)$$

where \mathfrak{F}_λ is the pointwise fixed subalgebra under the (restricted set of) automorphisms

$$\phi \rightarrow \phi + 2\pi n \lambda, \quad n \in \mathbb{Z}. \quad (7.261)$$

Thus, the algebra \mathfrak{F}_λ describes a free compactified real scalar field with compactification radius λ . The parameter λ has dimension $(d-2)/2$. We have that $\mathfrak{F}_\lambda \subset \mathfrak{F}_{\lambda'}$ if and only if λ' is an integer multiple of λ . According to (7.258), \mathfrak{F}_λ is generated by the Weyl operators

⁶¹Expressions (7.256-7.257) do not hold for unbounded regions nor for the whole spacetime, because in these cases, the integral (7.257) does not converge to a well-defined self-adjoint operator. However, for unbounded standard regions we can still construct local implementations of (7.255) [168]. This is no longer possible for the whole spacetime, i.e. there is no unitary global implementation of (7.255) [1].

⁶²For non-compact groups the group average (7.91) is, in principle, not well-defined. However, there is a subclass of non-compact groups, called *amenable groups*, where a sort of group average, called *mean*, could already be constructed. However, such a mean is far to be unique as in the compact case. We remark that any abelian group, and in particular \mathbb{R} , is amenable.

$e^{i\phi(f)}$ satisfying the constrain

$$\int_{\mathbb{R}^d} f(x) d^d x = \frac{k}{\lambda}, \quad \text{for some } k \in \mathbb{Z}. \quad (7.262)$$

The local algebras of \mathfrak{F}_λ can be defined in the same way as for the algebra \mathcal{A} , but restricting the smearing functions to the ones satisfying (7.259). The inclusion $\mathfrak{F}_\lambda \subsetneq \mathfrak{F}$ behaves in similar way as the one $\mathcal{A} \subsetneq \mathfrak{F}$ above. Here, we have again that the local algebras attached to double cones are not maximal in the vacuum representation, where \mathfrak{F}_λ acts irreducible. The dual net corresponding to \mathfrak{F}_λ is also \mathfrak{F} . However, in this case the symmetry group $\mathfrak{F} \rightarrow \mathfrak{F}_\lambda$ is \mathbb{Z} , which is also spontaneously broken.

Here we are interested in the inclusion $\mathcal{A} \subsetneq \mathfrak{F}_\lambda$. The algebra \mathcal{A} is the pointwise fixed subalgebra under the automorphisms of \mathfrak{F}_λ

$$\phi \rightarrow \phi + k\lambda, \quad k \in [-\pi, \pi]. \quad (7.263)$$

Therefore, there is a $G := U(1)$ spontaneously broken symmetry between \mathfrak{F}_λ and \mathcal{A} . Let W be a double cone and define

$$V := e^{i\lambda^{-1}\phi(f)}, \quad \text{supp}(f) \subset W, \quad \int_{\mathbb{R}^d} d^d x f(x) = 1, \quad (7.264)$$

which belongs to $\mathfrak{F}_\lambda(W)$ but not to $\mathcal{A}(W)$. We have that $\mathfrak{F}_\lambda(W) = \mathcal{A}(W) \vee \{V_W\}$.⁶³ Thereby, $\{V^n : n \in \mathbb{Z}\}$ is an Abelian subalgebra of $\mathfrak{F}_\lambda(W)$ which is not included in $\mathcal{A}(W)$. In fact, such an Abelian subalgebra is the main difference between $\mathcal{A}(W)$ and $\mathfrak{F}_\lambda(W)$. The expectation values of charged operators in \mathfrak{F}_λ , with respect to the $U(1)$ -symmetry, have non-zero expectation values in the vacuum state

$$\langle V^n \rangle = e^{-\frac{n^2}{2\lambda^2} \langle f \cdot C \cdot f \rangle}, \quad (7.265)$$

where we have defined

$$f_1 \cdot C \cdot f_2 := \int_{\mathbb{R}^{2d}} d^d x d^d y f_1(x) C(x-y) f_2(y),$$

$$C(x) := \langle \phi(x) \phi(0) \rangle = \frac{\Gamma(\Delta)}{4\pi^{\frac{d}{2}}} \frac{1}{(x - i0^+ e_0)^{2\Delta}}, \quad \Delta := \frac{d-2}{2}, \quad e_0 := (1, \bar{0}). \quad (7.266)$$

Now we consider two double cones $W_1 \times W_2$ and we define $\mathcal{I}_{12} := V_1 V_2^\dagger$, where $V_j \in \mathfrak{F}_\lambda(W_j)$ are as in (7.264). The operator \mathcal{I}_{12} belongs to the algebra \mathcal{A} , commutes with all operators in $\mathfrak{F}_\lambda((W_1 \vee W_2)')$, but it does not belong to the additive algebra $\mathcal{A}(W_1 \vee W_2)$. Hence, the operator \mathcal{I}_{12} is a kind of intertwiner, but not related to any SS.

⁶³We also have that $\mathfrak{F}_\lambda = \mathcal{A} \vee \{V_W\}$ for any operator V_W as in (7.264) and any double cone W .

Let us now study the entropic order parameter (7.240) for a single region. For that, we consider the standard double cone $W := D(\mathcal{C}_R)$. We can estimate $S_{\mathfrak{F}_\lambda(W)}(\omega|\omega \circ \varepsilon)$ computing the RE on the subalgebra $\{V^n : n \in \mathbb{Z}\}$, maximizing over all smearing functions satisfying (7.264). As happens with the intertwiner algebra for a $U(1)$ -symmetry in section 7.2.5, this subalgebra corresponds to the classical probability space of probability distributions over the unit circle S^1 . The state $\omega \circ \varepsilon$ corresponds to the the constant probability density $p_{\omega \circ \varepsilon}(k) := (2\pi)^{-1}$. The other state depends on the vacuum expectation values of V_W^n (see equation (7.265))

$$p_\omega(k) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{ikn} \langle V^n \rangle, \quad k \in [-\pi, \pi]. \quad (7.267)$$

We have to take wide smearing functions to get the maximal RE. In analogy with the computation in section 7.2.5, if the coefficient $(f \cdot C \cdot f)/(2\lambda^2)$ multiplying n^2 in the exponent of (7.265) is small, we get a RE

$$S_{\mathfrak{F}_\lambda(W)}(\omega|\omega \circ \varepsilon) \simeq -1/2 \log((f \cdot C \cdot f)/\lambda^2). \quad (7.268)$$

This is the case when $R\mu \gg 1$, where we have defined $\mu := \lambda^{\frac{2}{d-2}}$ to the energy scale of λ . On the other hand, for small R , the coefficient of the exponent in (7.265) is large and the probability is concentrated around $n = 0$ as for $\omega \circ \varepsilon$. The RE has a change of regime at $R\mu \sim 1$ and goes to zero for $R\mu \rightarrow 0^+$ as happens for finite groups. Then, we get

$$S_{\mathfrak{F}_\lambda(W)}(\omega|\omega \circ \varepsilon) \simeq \begin{cases} \frac{(d-2)}{2} \log(R\mu), & R\mu \gg 1, \\ 0, & R\mu \ll 1. \end{cases} \quad (7.269)$$

We now study the MI difference between \mathfrak{F}_λ and \mathcal{A} , which can be investigated using the same tools developed so far. Then, we consider the regions $W_1 := D(\mathcal{C}_R)$ and $W_2 := D(\mathcal{C}'_{R+\epsilon})$ ($\epsilon > 0$). For W_2 unbounded, the first and third terms on the r.h.s. of (7.242) independently diverge, but its difference keeps finite. For the calculations below, it is useful to take W_2 large but bounded, to maintain both terms independently finite, and take the limit of $W_2 \rightarrow D(\mathcal{C}'_{R+\epsilon})$ in the end.

To evaluate the MI difference let us first investigate the contribution of the intertwiners. This model has the nice feature that we can explicitly compute their expectation values. We form a subalgebra of intertwiners using the integer powers of the one mode $(V_1 V_2^\dagger)^n$. This Abelian subalgebra \mathcal{B}_{12} corresponds to the classical probability space of probability distributions over the unit circle S^1 . The expectation values are

$$\langle V_1^n V_2^{-n} \rangle = e^{-\frac{n^2}{2} \sigma^2}, \quad (7.270)$$

with

$$\sigma^2 := \frac{1}{\lambda^2}(f_1 \cdot C \cdot f_1 + f_2 \cdot C \cdot f_2 - 2 f_1 \cdot C \cdot f_2). \quad (7.271)$$

This has the general Gaussian form of (7.191), but here the expression is exact. Again, this gives, for small $\sigma^2 \ll 1$ and through a Fourier transform, a Gaussian classical probability distribution $p_\omega(k)$. The state $\omega \circ \varepsilon_{12}$ corresponds to a uniform classical probability density $p_{\omega \circ \varepsilon_{12}}(k) := (2\pi)^{-1}$. The RE is given by the classical Kullback–Leibler divergence

$$S_{\mathcal{B}_{12}}(\omega|\omega \circ \varepsilon_{12}) = H_{KL}(p_\omega|p_{\omega \circ \varepsilon_{12}}) = -\log(\sigma). \quad (7.272)$$

In order to minimize σ in (7.271), α_1 and α_2 have to be near to each other lying along the boundary. The minimization depends on the geometry, more precisely, on the total area available $A \sim R^{d-2}$ and the separation distance ϵ , but is independent of the compactification radius λ . Then, we can use symmetric test functions approximately translational invariant along the boundary surface to get

$$\sigma^2 \simeq (\lambda^2 R^{d-2} f(R/\epsilon))^{-1}. \quad (7.273)$$

The area factor within the brackets is dictated by dimensional reasons and the extensivity of the problem along the entanglement surface. The factor $f(R/\epsilon)$ should be a slowly varying function resulting from the minimization in the shape of the test functions in the direction perpendicular to the boundary. This factor should ensure $\sigma \rightarrow 0$ whenever $\epsilon \rightarrow 0^+$, though at a slow pace. What we want to emphasize is that this cannot be further improved to be of the order $(\epsilon/R)^{d-2}$ as in the case of a the $U(1)$ -symmetry with a conformal current in the UV studied in section 7.2.5. It can also be checked that it cannot be improved by taking a larger algebra formed by charge creating operators for different modes along the surface.

The twist version tells a parallel story but it is easier to compute the dependence on ϵ . The twists can be constructed with integrals of the current $J^0 = \partial_0 \phi$ as in (7.256–7.257). Its expectation values are

$$\langle \tau_k \rangle = e^{-\frac{1}{2} \lambda^2 k^2 (f \cdot C_J \cdot f)}, \quad (7.274)$$

where $C_J(x) := -\partial_0^2 C(x)$. As in section 7.2.5, these expectation values (and the twists (7.257) itself) are very good approximations for an exponent which is large around $|k| \gtrsim \pi$, because the result is not actually periodic $k \rightarrow k + 2\pi$. However, it is important that these expectation values are interpreted in terms of discrete probabilities of a classical random variable taking values $q \in \mathbb{Z}$, and the twist expectation values are represented as $\langle \tau_k \rangle = \sum_{q \in \mathbb{Z}} e^{iqk} p_q$. Following the same reasoning as in section 7.2.5, the twist subalgebra entropy is

$$S_{G_\tau}(\omega) \simeq \frac{1}{2} \log(\lambda^2 (f \cdot C_J \cdot f)). \quad (7.275)$$

To estimate the argument of the logarithm and avoid integrals over coinciding points we can use the fact that the twist belongs to the neutral algebra \mathcal{A} and the vacuum state on the neutral algebra is invariant under the symmetry transformations (7.263). A similar computation as the one performed in subsection 7.3.4 gives

$$f \cdot C_J \cdot f \simeq R^{d-2} \log(R/\epsilon), \quad \frac{R}{\epsilon} \gg 1. \quad (7.276)$$

Hence, we expect

$$S_{G_\tau}(\omega) \simeq \frac{d-2}{2} \log(\mu R) + \frac{1}{2} \log(\log(R/\epsilon)), \quad \frac{R}{\epsilon} \gg 1, R\mu \gg 1, \quad (7.277)$$

independently of the value of $\epsilon\mu$. This is similar to the case of a general $U(1)$ -symmetry with a conformal current in the UV given by (7.193), but the cutoff has been replaced by the scale of compactification. The dependence on the cutoff is subleading but still divergent as it must be, since the size of the group is infinite. The reason of the difference with (7.193) is clearly that the conserved current $J_\mu(x)$ is not conformal, it has dimension $d/2$ instead of $d-1$, giving a much smaller charge fluctuation rate for short distances. The result (7.277) also holds for R smaller than the compactification scale provided the argument in the logarithm in (7.275) is still large, or equivalently, (7.277) is still positive. This curiously seems to require very small ϵ as we decrease the radius. For small R and fix ϵ , the intertwiner expectation value is very concentrated around the identity, i.e. around $n=0$. We cannot use a continuous charge approximation to get the probability distribution $p_\omega(k)$, and these probabilities are given by a Fourier series with coefficients proportional to (7.270) as in (7.267). Then, the RE goes to zero fast with $R\mu \rightarrow 0^+$.

Coming back to the MI difference, we can follow the same reasoning as above for finite groups. For W_2 bounded the result is finite and then we take the limit of a large region W_2 with the rest of the geometry fixed. Then, the last term in (7.242) should be canceled by similar contribution coming from the first term, which is given by the same charged fields as the ones contributing to the last term. After this cancellation, there is a competition between the intertwiner and a charged operator in W_1 to the first term. For ϵ small enough, as corresponds to a cutoff, the intertwiner dominates and we should have

$$\Delta I \simeq \frac{d-2}{2} \log(\mu R) + \frac{1}{2} \log(\log(R/\epsilon)) - S_{\mathfrak{F}_\lambda(W_1)}(\omega|\omega \circ \varepsilon), \quad \epsilon \ll R, \mu^{-1}. \quad (7.278)$$

Then, replacing (7.269) into (7.278), we get, for $\epsilon \ll R, \mu^{-1}$,

$$\Delta I \simeq \begin{cases} \frac{d-2}{2} \log(\mu R) + \frac{1}{2} \log(\log(R/\epsilon)), & R\mu \ll 1, \\ \frac{1}{2} \log(\log(R/\epsilon)), & R\mu \gg 1. \end{cases} \quad (7.279)$$

For small $R\mu \ll 1$, we note that the negative sign of the first term for small $R\mu$ must be supported by a compensating sign of the second term, and we need an exponentially small cutoff, as already remarked previously. For large $R\mu \gg 1$, ΔI does not vanish, in contrast to the case of a finite group. It has the same dependence on ϵ as in the UV. On the other hand, for large ϵ , we expect ΔI to vanish in the regime of small $R\mu$.

7.2.8.2. SSB of a compact Lie group

Now we consider the case of a compact Lie symmetry group spontaneously broken with a conformal current in the UV. Let us first discuss the order parameter $S_{\mathfrak{F}(W)}(\omega|\omega \circ \varepsilon)$ for the double cone $W := D(\mathcal{C}_R)$. For large R , we can compute this quantity with the RE on the algebra of compactified scalars that are the Goldstone modes. As in the previous discussion, we get

$$S_{\mathfrak{F}(W)}(\omega|\omega \circ \varepsilon) \simeq \frac{\mathcal{G}(d-2)}{2} \log(R\mu), \quad (7.280)$$

where μ is the SSB scale that is taken of the same order as the compactification radius, and \mathcal{G} is the number of Goldstone bosons. For smaller radius, we cannot use the Goldstone boson approximation any more, but we expect that the RE has a change of regime at $R\mu \sim 1$ and goes to zero for $R\mu \rightarrow 0^+$, as happens for finite groups and compact scalars.

For the MI difference, we can follow the same reasoning as above for the compact scalar and finite groups. We take spacetime regions $W_1 := D(\mathcal{C}_R)$ and $W_2 := D(\mathcal{C}'_{R+\epsilon})$ ($\epsilon > 0$). The intertwiners dominate the rest of the contribution to the first term of (7.242) and, using the results on section 7.2.5, we have

$$\Delta I \simeq \frac{\mathcal{G}(d-2)}{2} \log(R/\epsilon) - S_{\mathfrak{F}(W_1)}(\omega|\omega \circ \varepsilon). \quad (7.281)$$

Then, according to the preceding discussion, we get, for $\epsilon \ll R, \mu^{-1}$,

$$\Delta I \simeq \begin{cases} \frac{\mathcal{G}(d-2)}{2} \log(R/\epsilon), & R\mu \ll 1, \\ -\frac{\mathcal{G}(d-2)}{2} \log(\epsilon\mu), & R\mu \gg 1. \end{cases} \quad (7.282)$$

This does not vanish for $R\mu \gg 1$, in contrast to the case of a finite group. It has the same dependence on ϵ as in the UV, but the dependence on the radius has been replaced by the SSB scale. Note that both models contain the massless scalar contribution of the Goldstone modes in the IR on top of this difference.

7.2.8.3. Remarks on previous results in the literature

We make some comments on previous results in the literature. In [170] there is a numerical study of the EE of a free Maxwell field in $d = 3$. This model is equivalent to the

algebra of derivatives of a free massless real scalar through the identification $\varepsilon_{\mu\nu\delta}F^{\nu\delta} = \partial_\mu\phi$. Then, the relation between the scalar and the Maxwell models is the same as between the scalar and the algebra of its derivatives. The symmetry $\phi \rightarrow \phi + s$ is uncompactified and the model does not contain any scales. This is equivalent to the model \mathfrak{F}_λ in the decompactification limit $\lambda \rightarrow +\infty$. Because of that, the RE $S_{\mathfrak{F}(W)}(\omega|\omega \circ \varepsilon)$ is not finite and we get the divergent quantity $\frac{1}{2} \log(R\mu)$ as $\mu = \lambda^{\frac{2}{d-2}} \rightarrow +\infty$. In the presence of a short distance cutoff $\delta > 0$ and with the naive lattice interpretation of the difference in entropies between the two models in place of the RE (see the discussion at the end of section 7.2.1), the compactification radius should get trade off by the cutoff, and we get, up to lattice ambiguities,

$$S_{\text{Maxwell}}(R) - S_{\text{scalar}}(R) \simeq \frac{1}{2} \log(R/\delta). \quad (7.283)$$

This is what was found numerically in [170]. For the MI difference, according to (7.279), in the limit of small ϵ and $\mu \rightarrow +\infty$, we get

$$I_{\text{scalar}}(W_1, W_2) - I_{\text{Maxwell}}(W_1, W_2) \simeq \frac{1}{2} \log(\log(R/\epsilon)). \quad (7.284)$$

This does not contain a $\log(R/\epsilon)$ term, and hence, it does not reproduce the difference in lattice entropies (7.283), which would have given the contradictory result that the MI of the smaller model would have been bigger than the one of the larger model. We checked numerically in the lattice, following the methods in [170], this dependence of the MI difference in the short ϵ limit, and we found agreement with (7.284). This term should be attributed as a negative contribution $-1/2 \log(\log(R\epsilon))$ to the Maxwell field MI rather than a positive one to the the scalar, which has a finite constant term. An analogous result is expected between the scalar and its derivatives (dual to higher form gauge fields) in any dimensions. Notice that the MI of the free scalar \mathfrak{F} is finite as well as the one of \mathcal{A} , even if there is a non-compact SSB relating it to \mathcal{A} . This is different from what we expect for a non-compact symmetry that is not spontaneously broken.

In [171] the authors study the change in entropy, between a free real scalar field \mathfrak{F} and a free real compact scalar \mathfrak{F}_λ , using the replica trick. For the MI, this is given by the subtraction of results in (7.279) with (7.284). Then, we get

$$I_{\mathfrak{F}_\lambda}(W_1, W_2) - I_{\mathfrak{F}}(W_1, W_2) = \begin{cases} \frac{d-2}{2} \log(\mu R), & R\mu \ll 1, \\ 0, & R\mu \gg 1. \end{cases} \quad (7.285)$$

This coincides with the result of [171] for the difference in entropies. Then, using $\Delta I/2$ as a proper renormalized entropy, our result differs from the one in [171] by a factor 1/2. This factor is typical of the difference between EE and $\Delta I/2$ for classically correlated variables.

The EE in models with SSB of continuous symmetries is also discussed in [172]. The

authors compute the EE in the case of SSB for a finite volume space. At finite size, symmetry is restored and the physics is the one of \mathfrak{F} , but with the symmetric state, which does not satisfy clustering in the large volume limit. They show that the EE for a large region of size R contains precisely a term of the form (7.280) as long as $R\mu \gg 1$. This explains previous lattice simulations [173]. This also coincides with the results from our calculations, since the lattice expression for the RE in (7.280) is

$$S_{\mathfrak{F}}(\omega \circ \varepsilon) - S_{\mathfrak{F}}(\omega) \sim \frac{\mathcal{G}(d-2)}{2} \log(R\mu), \quad R\mu \gg 1. \quad (7.286)$$

The symmetric state has a new term that shows up in the calculations of [172]. Note that the MI difference has a completely different behavior.

7.2.9. The case of $d = 2$ and chiral CFTs

For the purpose of this section let us consider the algebras attached to spacelike open regions over the Cauchy surface $\Sigma_0 \cong \mathbb{R}$ at time $x^0 = 0$. For the case of a chiral CFT, the algebras are attached to open regions over the null line $\mathcal{N} \cong \mathbb{R}$. In both cases, the topology of the space is \mathbb{R} . The double cones corresponds to bounded open intervals $A \subset \mathbb{R}$.

The main reason of why the theory of DHR SS has significant differences in this case with respect to the case $d \geq 3$ is that the complement of double cone is a disconnected region formed by two semi-infinite regions. This gives a non-trivial statistics for the charged sectors. This implies that the SS structure does not necessarily come from a compact symmetry group, and it has to be described more generally by their statistical dimensions and fusion rules.

Apart from that, a second difference arises when we compute the MI between a double cone W_1 and a nearly complementary region W_2 . In this case, the shell $S := (W_1 \vee W_2)'$ is a disconnected region formed by two bounded intervals of length $\epsilon \gtrsim 0$, whereas for $d \geq 3$ this region is connected. Let still consider the case where \mathcal{A} is the pointwise fixed subalgebra of \mathfrak{F} under a finite symmetry group G . For a massive theory, we get twice the value corresponding to higher dimensions

$$\Delta I = 2 \log |G|. \quad (7.287)$$

This can be thought as an instance of (7.211) since the shell and W_2 are disconnected regions, and hence, there are two sets of independent intertwiners connecting W_1 with the two connected components of W_2 . However, if the theory is conformal, W_2 can be thought of as a single interval because the theory is defined over the unit circle $S^1 \cong \overline{\mathbb{R}}$. In this case, we obtain

$$\Delta I = \log |G|. \quad (7.288)$$

If the two regions W_1 and W_2 touch each other on one side and they stay enough separated from the other side, we get $\Delta I = \log |G|$ in both cases, conformal and massive.

A case which can be computed exactly is the free chiral current that we have studied in chapter 6. By bosonization, this is the same model as the one obtained by restricting the algebra \mathfrak{F} , of a free chiral fermion field, to the subalgebra \mathcal{A} generated by the fermionic current $:\psi^\dagger(x)\psi(x):$. The group symmetry is $G = U(1)$. As both theories are conformally invariant, their MIs on the vacuum state are only functions of the cross-ratio

$$\eta := \frac{(b_1 - a_1)(b_2 - a_2)}{(a_2 - a_1)(b_2 - b_1)} \in (0, 1), \quad (7.289)$$

which depends on the endpoints of the intervals $A_1 := (a_1, b_1)$ and $A_2 := (a_2, b_2)$. Using the results of chapter 6, we write⁶⁴

$$I_{\mathfrak{F}}(\eta) = -\frac{1}{6} \log(1 - \eta), \quad (7.290)$$

$$I_{\mathcal{A}}(\eta) = -\frac{1}{6} \log(1 - \eta) - \tilde{U}(\eta), \quad (7.291)$$

where $\tilde{U}(\eta) \geq 0$.⁶⁵ Using the results of (6.3.2.4) for the present set up ($b_1 - a_1 =: R$, $a_2 = b_1 + \epsilon$ and $b_2 = a_1 - \epsilon$), we obtain that the MI difference $\Delta I(\eta) = \tilde{U}(\eta)$ behaves as

$$\Delta I(\eta) \sim \frac{1}{2} \log(-\log(1 - \eta)) \sim \frac{1}{2} \log(\log(R/\epsilon)), \quad \epsilon \rightarrow 0^+, \quad (7.292)$$

where $\eta \sim 1 - (\epsilon/R)^2$ and R is the size of the interval corresponding to the region A_1 .

Assuming the usual relation that the entropy for complementary regions is the same in a pure global state, we would have

$$S(I_1 \cup I_3) = S(I_2 \cup I_4), \quad (7.293)$$

where we are thinking in a compactified real line $S^1 \cong \overline{\mathbb{R}}$ divided in four intervals. Completing this relation with the entropies of the single intervals to get MIs,⁶⁶ for a CFT in $d = 2$, (7.293) translates into the symmetry relation [26]

$$I(\eta) = I(1 - \eta) - \frac{c}{6} \log\left(\frac{1 - \eta}{\eta}\right), \quad (7.294)$$

where c is the central charge (summed over both chiralities). This symmetry property does not hold when there are superselection sectors which ruin (7.293), and in particular is badly broken for the chiral current (not for the chiral fermion) as we have shown in the

⁶⁴For the MI of the chiral fermion see also [36, 62, 133]

⁶⁵ $\tilde{U}(\eta)$ is called $-U(\eta)$ in (6.215).

⁶⁶The single interval entropies are $S(r) = (c/6) \log(r/\epsilon) + \text{const.}$, where r is the size of the interval.

previous chapter. This symmetry leads to $\tilde{U}(\eta) = \tilde{U}(1 - \eta)$, while for the current we have $\tilde{U}(0) = 0$, because large distance mutual information vanishes, and $\tilde{U}(1) = +\infty$. For the case of a finite symmetry group, $\Delta I(\eta = 0) = 0$ while $\Delta I(\eta = 1) = \log |G|$.

Relation (7.294) can be shown from the replica trick using modular invariance for non-chiral models [126]. In connection to this, it has been shown more generally that modular invariant models are complete (duality holds for two intervals) and the symmetry property (7.294) holds [150].

It is to be noted that in $d = 2$ it does not hold any more $\Delta I = S_{\mathcal{A}'_{34}}(\omega|\omega \circ \varepsilon)$, where $\varepsilon : \mathcal{A}'_{34} \rightarrow \mathcal{A}_{12}$. This is because, in $d = 2$, the algebra \mathcal{A}'_{34} , in addition to the intertwiners, contains the twists, which are not in \mathfrak{F}_{12} . It is expected that $S_{\mathcal{A}'_{34}}(\omega|\omega \circ \varepsilon)$ has limit $2 \log |G|$ at the point of contact, instead of $\log |G|$ as happens for ΔI .

The case of general SS not necessarily coming from a symmetry group was treated in [62, 150, 174]. Each sector has a statistical dimension $d_r \geq 1$, which can be a non-integer and it can only take some specific values for $d_r < 2$ [43]. Generalizing the result we got for a group, we have, for the limit of contact between the complementary intervals, in a $2d$ CFT

$$S_{\mathcal{A}'_{34}}(\omega|\omega \circ \varepsilon) = \log \mathcal{D}^2 = \log \sum_r d_r^2, \quad (7.295)$$

where the sum runs over the irreducible sectors, and $\mathcal{D}^2 := \sum_r d_r^2$ is called the *quantum dimension* of the model \mathcal{A} . In terms of the quantum dimensions of the SS, this formula is the same in any spacetime dimension. It is also the index of inclusion of algebras $\mathcal{A}_{12} \subset \mathcal{A}'_{34}$ [43, 72, 137]. The result (7.295) holds for finite index.

7.3. Examples of intertwiner and twist bounds

In this section, we compute in some concrete examples, the intertwiner and twists expectation values. The objective is to build up intuition on how they generally behave in order to provide the best bounds available. We describe how intertwiners are assimilated to edge modes localized near the boundary and how they tend to minimize the modular energy for nearly complementary regions, while they spread out for distant regions. We also describe how the charge creating operators in the same region try to minimize their mutual entanglement, and hence, to repel each other. Finally, concerning the squeezed twists, we show that, in the short ϵ limit, they are exponentially suppressed by the area and have Gaussian expectation values for Lie groups.

7.3.1. Intertwiners at short distance

In this subsection, we explain, for the fermion field, how the unitary intertwiner \mathcal{I}_{12} of two complementary regions can be suitably chosen in order to maximize its vacuum expectation value, i.e. $|\langle \mathcal{I}_{12} \rangle| \simeq 1$.⁶⁷ We consider the theory of a free fermion field and the \mathbb{Z}_2 bosonic symmetry discussed in section 7.1.1. An intertwiner between a region W_1 and its complement $W_2 := W_1'$ can be written as

$$\mathcal{I}_{12} := V_1 V_2^\dagger, \quad (7.296)$$

where $V_i \in \mathfrak{F}(W_i)$ are fermionic unitary operators made out of the fermion fields as in (7.10) and (7.47)

$$V_j := \psi(f_j) + \psi^\dagger(f_j), \quad (7.297)$$

where f_j are spinor valued functions supported in the regions W_j . Here we use the description at fixed time x^0 . The spacetime regions $W_j := D(\mathcal{C}_j)$ are the Cauchy development of the complementary space regions $\mathcal{C}_1 \subset \mathbb{R}^{d-1}$ and $\mathcal{C}_2 = \mathbb{R}^{d-1} - \mathcal{C}_1$. Then, we write the smearing functions as $f_j(x) := \delta(x^0) \alpha_j(\bar{x})$. Expression (7.297) automatically gives $V_j = V_j^\dagger$. In order to have $V_j^{-1} = V_j^\dagger$ we must also impose

$$\int_{\mathcal{C}_j} d^{d-1}x \alpha_j^\dagger(\bar{x}) \alpha_j(\bar{x}) = 1. \quad (7.298)$$

Using that the local field algebra $\mathfrak{F}(W_1)$ satisfies duality,⁶⁸ we can choose the unitary $V_2 \in \mathfrak{F}(W_2)$ as the vacuum modular conjugated of some unitary operator $\tilde{V}_1 \in \mathfrak{F}(W_1)$

$$V_2 =: -i(ZJ)^\dagger \tilde{V}_1^\dagger ZJ, \quad (7.299)$$

where Z is the operator defined in definition 2.68, constructed from the grading of the algebra and it takes account that the algebras of W_1 and W_2 graded commute.⁶⁹ Since modular conjugation respects statistics, V_2 is a fermionic operator iff \tilde{V}_1 is fermionic.⁷⁰ Using the relations for the modular operator and modular conjugation for a fermionic model [175], we can rewrite the vacuum expectation value of the intertwiner (7.296) as

$$\langle V_1 V_2^\dagger \rangle = \langle V_1 \Delta^{\frac{1}{2}} \tilde{V}_1^\dagger \rangle = \langle V_1 e^{-\frac{1}{2}K} \tilde{V}_1^\dagger \rangle, \quad (7.300)$$

⁶⁷We always have that $|\langle \mathcal{I} \rangle| \leq 1$, since \mathcal{I} is unitary.

⁶⁸This for sure happens for double cones because our assumption. According to Bisognano-Wichmann theorem it also happens for Wedges. Moreover, according to our discussion at the end of section 7.1.2, duality should also hold for two double cones, since the field algebra has no DHR SS.

⁶⁹For fermionic nets, the modular conjugation J must be replaced by the twisted modular conjugation ZJ [175]. Regardless this technicality, the outcome of the argument below holds.

⁷⁰Moreover, for free fields, V_2 is of the form (7.297) if \tilde{V}_1 is of that form too.

where K is the “full” vacuum modular Hamiltonian of the algebra $\mathfrak{F}(W_1)$. We can write it in terms of the inner modular Hamiltonians $K := K_{W_1} - K_{W'_1}$. Moreover, for the purpose of the expression (7.300), we need only to know K_{W_1} since $V_1, \tilde{V}_1 \in \mathfrak{F}(W_1)$. To lighten the notation, we make an abuse of notation and denote $K_1 := K_{W_1}$.

Expression (7.300) indicates that we can search for a maximum in the expectation value within the choices $\tilde{V}_1 = V_1$. Here we invoke all the relations developed in sections 4.3.1 and 4.3.4. The “inner” modular Hamiltonian is quadratic in the field operator and the eigenfunctions $u_{s,k}(\bar{x})$ of the modular Hamiltonian kernel are the same as those of the correlator kernel (see equations from (4.92) to (4.98)). To compute (7.300) is useful to use the operators defined in equation (4.102), namely

$$\tilde{\psi}(s, k) := \int_{\mathcal{C}_1} d^{d-1}x u_{s,k}^\dagger(\bar{x}) \psi(\bar{x}), \quad \psi(x) = \sum_k \int_{-\infty}^{+\infty} ds u_{s,k}(\bar{x}) \tilde{\psi}(s, k), \quad (7.301)$$

which satisfy the CAR relations (4.103). Replacing (7.301) into (4.97), we show that the modular Hamiltonian is diagonal in these new modes

$$K_1 = \sum_{k \in \Upsilon} \int_{-\infty}^{+\infty} ds \tilde{\psi}(s, k)^\dagger 2\pi s \tilde{\psi}(s, k). \quad (7.302)$$

Using (7.301), we can rewrite (7.297) as⁷¹

$$V := \sum_{k \in \Upsilon} \int_{-\infty}^{+\infty} ds \left[\tilde{\psi}(s, k) \tilde{\alpha}(s, k)^* + \tilde{\psi}^\dagger(s, k) \tilde{\alpha}(s, k) \right], \quad (7.303)$$

where

$$\tilde{\alpha}(s, k) := \int_{\mathcal{C}} d^{d-1}x u_{s,k}^\dagger(\bar{x}) \alpha_1(\bar{x}), \quad (7.304)$$

and the normalization condition (7.298) implies

$$\sum_{k \in \Upsilon} \int_{-\infty}^{+\infty} ds |\tilde{\alpha}(s, k)|^2 = 1. \quad (7.305)$$

The vacuum correlators for these new modes can be easily obtained from (4.93) and (7.301)

$$\left\langle \tilde{\psi}^\dagger(s, k) \tilde{\psi}(s', k') \right\rangle = \frac{1}{1 + e^{2\pi s}} \delta_{kk'} \delta(s - s'). \quad (7.306)$$

Replacing (7.302) and (7.303) into (7.300), and using (7.306) and the fact that

$$[K, V_1] = 0 \quad \text{and} \quad K|0\rangle = 0, \quad (7.307)$$

⁷¹From now on, we omit the subscripts j in V_j , α_j and \mathcal{C}_j .

a straightforward computations gives

$$\langle V_1 V_2^\dagger \rangle = \langle V_1 \Delta^{\frac{1}{2}} V_1^\dagger \rangle = \sum_{k \in \Upsilon} \int_{-\infty}^{+\infty} \frac{|\tilde{\alpha}(s, k)|^2}{\cosh(\pi s)} ds. \tag{7.308}$$

It is not possible to choose a charge creating operator that commutes with the modular Hamiltonian, which would imply $\langle V_1 V_2^\dagger \rangle = 1$ exactly, because it is charged and the modular Hamiltonian is neutral.⁷² But we can choose a charged mode with a very small modular energy. As this formula clearly displays, in order to maximize the expectation value of the intertwiner we have to construct a wave packet with small modular energy by localizing $\tilde{\alpha}(s, k)$ sharply around $s = 0$. There is a lot of freedom in approaching this limit. For example, we can choose Gaussian wave packets

$$\tilde{\alpha}(s, k) := \frac{\sqrt{p_k}}{(2\pi)^{\frac{1}{4}} \sqrt{\sigma}} e^{-\frac{s^2}{4\sigma^2} - i\lambda_k s}, \tag{7.309}$$

with $\lambda_k \in \mathbb{R}$, $\sigma > 0$, $0 \leq p_k \leq 1$ and $\sum_{k \in \Upsilon} p_k = 1$. Under this choice, we have

$$\langle V_1 V_2^\dagger \rangle \xrightarrow{\sigma \rightarrow 0^+} 1. \tag{7.310}$$

As an example, we have $\langle V_1 V_2^\dagger \rangle > 0.99$ for $\sigma = \frac{1}{22}$. Notice that we still have the freedom to choose different phases and probabilities for the different values of the degeneracy parameter $k \in \Upsilon$. The values of the degeneracy parameters are in correspondence with the variables describing the boundary of the region [38].

After we have shown that the expectation value of the free fermion intertwiner can be (asymptotically) maximized, we want to see how these “maximized” wave packets are localized in position space. We expect that such wave packets are more and more supported around the boundary $\partial\mathcal{C}_1$ as long as the expectation value (7.310) approximates to 1. Certainly, the modular conjugated operator V_2 will be also located around $\partial\mathcal{C}_1$. In the following, we will show this behavior in some examples.

7.3.1.1. Free chiral fermion and Rindler wedge

We consider the right Rindler wedge $\mathcal{C}_1 := \{x > 0\}$. The normalized eigenfunctions are [39]

$$u_s(x) := \frac{e^{is \log(x)}}{\sqrt{2\pi x}}, \quad x > 0, \tag{7.311}$$

⁷²More precisely, the modular group Δ_Ω^{it} , of any invariant vector $|\Omega\rangle \in \mathcal{H}_0$ and any subalgebra $\mathfrak{F}(\mathcal{O})$, belongs to the neutral algebra G' . Then, the unbounded modular Hamiltonian K_Ω is *neutral* in the sense that all its bounded spectral projectors belongs to G' . In particular, we have that $\langle K_\Omega \Psi_1 | U(g) | \Psi_2 \rangle = \langle \Psi_1 | U(g) K_\Omega | \Psi_2 \rangle$ for all $|\Psi_1\rangle, |\Psi_2\rangle \in \text{Dom}(K_\Omega)$ and all $g \in G$.

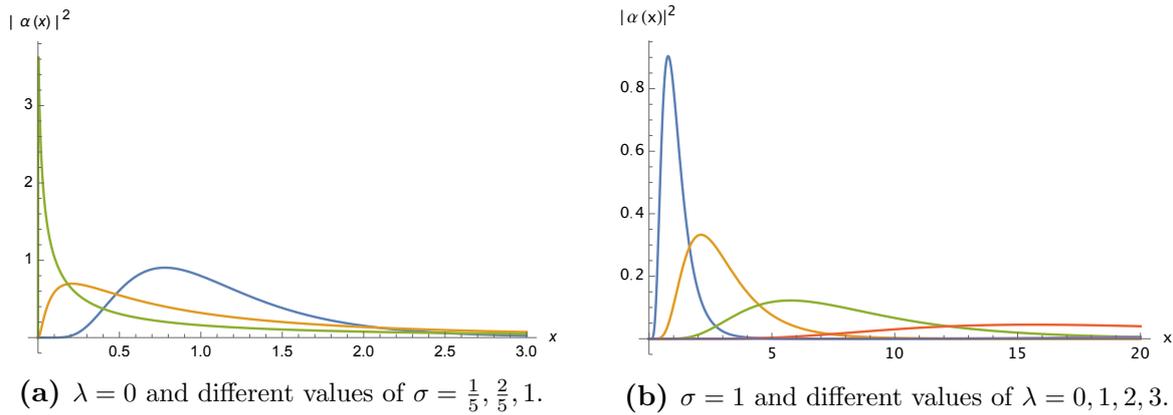


Figure 7.4: Localization of the wave packet $|\alpha(x)|^2$ (eq. (7.312)) for the free chiral fermion and the right Rindler wedge.

and the “modular” Fourier transform (7.304) can be analytically done

$$\alpha(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\sigma}{x}} e^{-\sigma^2(\log(x)-\lambda)^2}. \quad (7.312)$$

The probability density $|\alpha(x)|^2$ is an ordinary Gaussian wave packet but in the logarithmic variable $z := \log(x)$. In figure 7.4a, we plot the the wave packet (7.312) for a fixed λ and different values of σ . Similarly, in figure 7.4b, we plot the same wave packet for a fixed σ and different values of λ . As we can see from figure 7.4a, as long as $\sigma \rightarrow 0$, the wave packet $|\alpha(x)|^2$ concentrates around $x = 0$. The job of the parameter λ is to move the wave packet center of mass inside the region $x > 0$. For a given λ , the wave packet concentrates $\frac{1}{2}$ of the probability between $0 < x < e^\lambda$.

7.3.1.2. Free chiral fermion and one interval

We set $\mathcal{C}_1 := (a, b)$ an interval. In this case, the normalized eigenfunctions are the one computed in chapter 6

$$u_s(x) = \sqrt{\frac{z'(x)}{2\pi}} e^{isz(x)}, \quad a < x < b, \quad (7.313)$$

where $z(x) := \log\left(\frac{x-a}{b-x}\right)$.⁷³ Then, the integral (7.304) can be analytically done

$$\alpha(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{z'(x)} \sqrt{\sigma} e^{-\sigma^2(\lambda-z(x))^2}. \quad (7.314)$$

The probability density $|\alpha(x)|^2$ is a Gaussian wave packet in the variable z . In figure 7.5a we show the wave packet (7.314) for a fixed λ and different values of σ . As we can

⁷³In chapter 6, we have used the notation $\omega(x)$ for the function $z(x)$.

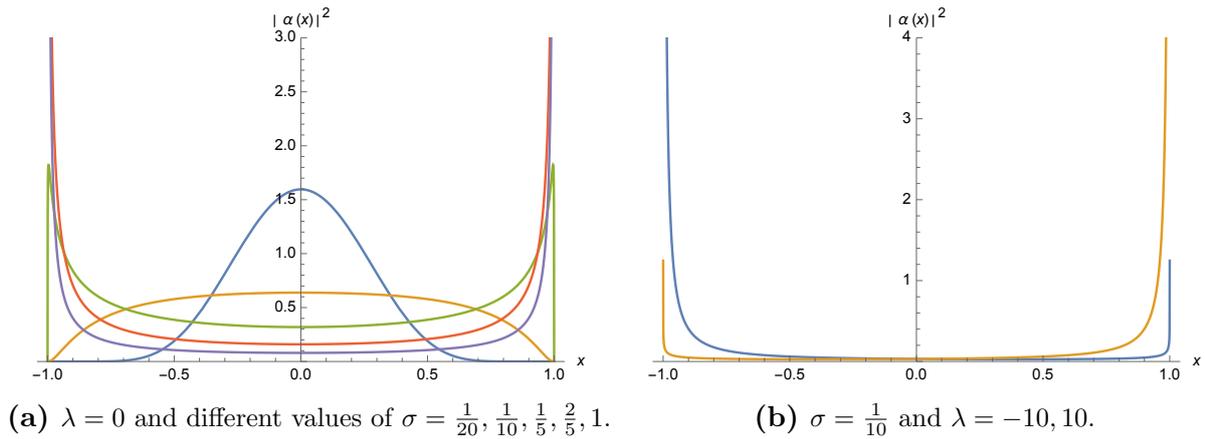


Figure 7.5: Localization of the wave packet $|\alpha(x)|^2$ (eq. (7.314)) for the free chiral fermion and a single interval. We set the interval endpoints to $a = -1$ and $b = 1$. The apparent divergencies when $x \rightarrow a^+, b^-$ are a matter of the plot resolution. The wave packet $|\alpha(x)|^2$ goes continuously to zero at the endpoints of the intervals.

appreciate, the support of the function $|\alpha(x)|^2$ concentrates around the endpoints a, b of the interval as long as $\sigma \rightarrow 0^+$.

For example, a straightforward computation, shows that, for $\lambda = 0$, the region $(a, a + \epsilon) \cup (b - \epsilon, b)$ with $\epsilon = (b - a) \left(1 + e^{\frac{1}{200\sigma}}\right)^{-1}$ concentrates $\gtrsim 0.99$ of the probability of $|\alpha(x)|^2$. In figure 7.5b, we plot the wave packet (7.314) for a fixed σ and two different values of λ . As we can see, the job of the parameter λ is to distribute the probability of the wave packet asymmetrically between the endpoints of the interval. In other words, we can freely choose the “maximizer” intertwiner to be located around $x = a$ or around $x = b$, or simultaneously around both endpoints with some relative probability that we can choose at our own will. To be more precise, we first redefine the real parameter λ as $\mu := \sqrt{2}\lambda\sigma$. Now, the probability distribution

$$|\alpha(x)|^2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} z'(x) \sigma e^{-(\mu - \sqrt{2}\sigma z(x))^2}, \quad (7.315)$$

has the limit

$$\lim_{\sigma \rightarrow 0^+} |\alpha(x)|^2 = q \cdot \delta(x - a) + (1 - q) \cdot \delta(x - b), \quad (7.316)$$

where $q = q(\mu) := \frac{1 - \text{erf}(\mu)}{2} \in [0, 1]$ and erf is the usual Gaussian distribution error function. If we combine different wave functions with different phases, we see that, as long as we are not interested in the precise form of the packet concentrated in the extremes of the interval, we have the freedom of a quantum mechanical wave function on a two dimensional Hilbert space seated at the two endpoints.

7.3.1.3. Free massive fermion in $d = 2$ and the Rindler wedge

We consider the right Rindler region $\mathcal{C}_1 := \{x > 0\}$. In this case, the theory does not decouple in chiralities and hence the eigenfunctions are two dimensional spinor functions. To compute them, we follow the same strategy of [38]⁷⁴ and we obtain

$$u_s(x) := \begin{pmatrix} u_{s,+}(x) \\ u_{s,-}(x) \end{pmatrix} = \frac{1}{\pi} \sqrt{m \cosh(\pi s)} \begin{pmatrix} K_{\frac{1}{2}-is}(mx) \\ -i K_{\frac{1}{2}+is}(mx) \end{pmatrix}, \quad (7.317)$$

where $K_\nu(z)$ is the modified Bessel function of 2nd kind. The probability of the a wave packet

$$\alpha(x) = \int_{-\infty}^{+\infty} ds \tilde{\alpha}(s) u_s(x), \quad (7.318)$$

to be localized around $x = 0$ is given by the behavior of the eigenfunctions in the limit $x \rightarrow 0$. We remember the limiting formulas

$$K_{\frac{1}{2}-is}(mx) \simeq \frac{1}{2} \Gamma\left(\frac{1}{2} - is\right) \left(\frac{mx}{2}\right)^{-\frac{1}{2}+is}, \quad (7.319)$$

$$K_{\frac{1}{2}+is}(mx) \simeq \frac{1}{2} \Gamma\left(\frac{1}{2} + is\right) \left(\frac{mx}{2}\right)^{-\frac{1}{2}-is}. \quad (7.320)$$

Then, for $x \simeq 0$, the eigenfunctions (7.317) become

$$u_{s,+}(x) \simeq \frac{1}{\sqrt{2\pi x}} e^{is \log\left(\frac{mx}{2}\right)}, \quad (7.321)$$

$$u_{s,-}(x) \simeq \frac{(-i)}{\sqrt{2\pi x}} e^{is \log\left(\frac{mx}{2}\right)}. \quad (7.322)$$

Then, for a Gaussian wave packet (7.309) we get

$$\alpha_+(x) = \int_{-\infty}^{+\infty} ds \tilde{\alpha}(s) u_{s,+}(x) \simeq \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\sigma}{x}} e^{-\sigma^2(\log\left(\frac{mx}{2}\right)-\lambda)^2}, \quad (7.323)$$

$$\alpha_-(x) = \int_{-\infty}^{+\infty} ds \tilde{\alpha}(s) u_{s,-}(x) \simeq (-i) \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\sigma}{x}} e^{-\sigma^2(\log\left(\frac{mx}{2}\right)+\lambda)^2}. \quad (7.324)$$

In other words, the probability density $|\alpha(x)|^2 = |\alpha_+(x)|^2 + |\alpha_-(x)|^2$ behaves, for wave packets localized near $x \simeq 0$, as the sum of two gaussian distributions in the variable $z := \log\left(\frac{mx}{2}\right)$. The analysis of the localization of such a wave packet can be followed from the massless case (subsection 7.3.1.1). We have to emphasize that expressions (7.323) and (7.324) are only valid for $x \simeq 0$. Then, only for $x_* \gtrsim 0$, the

⁷⁴We must first solve the Euclidean Dirac equation $(\gamma_\mu \partial_\mu + m) S_s(x, y) = 0$ for $(x, y) \in \mathbb{R}^2$ with a multiplicative boundary condition $S_s(x, 0^+) = -e^{2\pi s} S_s(x, 0^-)$ for $x > 0$ (the Rindler wedge region). Then, the eigenfunctions are $u_s(x) := c_s S_s(x, 0^+)$ with $c_s \in \mathbb{C}$ a normalization constant.

probability $\int_0^{x_*} (|\alpha_+(x)|^2 + |\alpha_-(x)|^2) dx$ can be calculated using expressions (7.323) and (7.324).

- For $\lambda \ll 0$, we have that $x_* = \frac{2}{m}e^\lambda \gtrsim 0$, and hence

$$p_+(0, x_*) = \int_0^{x_*} dx |\alpha_+(x)|^2 = \frac{1}{2}, \quad (7.325)$$

$$p_-(0, x_*) = \int_0^{x_*} dx |\alpha_-(x)|^2 \simeq 0. \quad (7.326)$$

- Conversely, for $\lambda \gg 0$, we have that $x_* = \frac{2}{m}e^{-\lambda} \gtrsim 0$, and then

$$p_+ = \int_0^{x_*} dx |\alpha_+(x)|^2 \simeq 0, \quad (7.327)$$

$$p_- = \int_0^{x_*} dx |\alpha_-(x)|^2 = \frac{1}{2}. \quad (7.328)$$

In other words, the free parameter λ determines which of the chiralities of the spinor wave function $\alpha(x)$ is localized around $x \simeq 0$. Since the full wave packet is normalized to 1, the other $\frac{1}{2}$ of the probability is distributed between both chiralities in order to have

$$p(x_*, +\infty) = \int_{x_*}^{+\infty} dx (|\alpha_+(x)|^2 + |\alpha_-(x)|^2) \simeq \frac{1}{2}. \quad (7.329)$$

Furthermore, the localization becomes sharper if we take into account the dependence of the probability with the dispersion σ . We recall that the above approximate wave packets (7.323) and (7.324) are only valid for a small dispersion $\sigma \rightarrow 0^+$ since they were calculated from the approximated eigenfunctions for $s \simeq 0$.

- For $\sigma \simeq 0$ and $\lambda \ll -\frac{1}{\sigma}$, we have that $x_* = \frac{2}{m}e^{\lambda + \frac{1}{\sigma}} \gtrsim 0$ and hence

$$p_+(0, x_*) = \int_0^{x_*} |\alpha_+(x)|^2 \gtrsim 0.977, \quad (7.330)$$

- For $\sigma \simeq 0$ and $\lambda \gg \frac{1}{\sigma}$, we now have that $x_* = \frac{2}{m}e^{-\lambda + \frac{1}{\sigma}} \gtrsim 0$, and then

$$p_-(0, x_*) = \int_0^{x_*} |\alpha_-(x)|^2 \gtrsim 0.977. \quad (7.331)$$

In other words, in the small limit $\sigma \rightarrow 0^+$ and for large positive (resp. negative) mean value λ , the positive (resp. negative) chirality of the wave packet $\alpha(x)$ is localized around the origin with probability very close to 1, while the negative (resp. positive) chirality has probability very close to 0 in the full region $x > 0$.

7.3.1.4. Free chiral fermion and n intervals

The multi-interval region is denoted as $\mathcal{C} := \bigcup_{j=1}^n (a_j, b_j)$, where $a_j < b_j < a_{j+1}$. According to our results of chapter 6, the eigenfunctions of the modular Hamiltonian are

$$u_s(x) := \frac{(-1)^{l+1}}{\sqrt{2\pi}} \frac{P(x)}{\sqrt{-\prod_{j=1}^n (x - a_j)(x - b_j)}} e^{isz(x)}, \quad x \in (a_l, b_l), \quad (7.332)$$

where

$$z(x) := \log \left(-\frac{\prod_{j=1}^n (x - a_j)}{\prod_{j=1}^n (x - b_j)} \right), \quad (7.333)$$

and $P(x)$ is a polynomial of degree $n - 1$. Moreover, we can choose n linearly independent polynomials P_k in order to form an orthonormal basis. As happened for the one interval case, for a wave packet in modular coordinates $\tilde{\alpha}(s, k)$ sharply localized around $s = 0$, the above eigenfunctions make the wave packet in position space to be highly localized around the intervals endpoints a_j, b_j . To be more precise, let us take a single normalized eigenfunction. Then, in analogy with the single interval case, we know that the wave function

$$\varphi(x) := \sqrt{z'(x)} \int_{-\infty}^{+\infty} ds \tilde{\alpha}(s) e^{isz(x)} \quad (7.334)$$

is localized in such a way that

$$|\varphi(x)|^2 = \sum_{l=1}^n [q \delta(x - a_l) + (1 - q) \delta(x - b_l)], \quad (7.335)$$

in the limit when the support of $\tilde{\alpha}(s)$ shrinks to $s = 0$.⁷⁵ The probability $q \in [0, 1]$ above can be freely chosen. Then, the wave packet becomes

$$\begin{aligned} \alpha(x) &= \int_{-\infty}^{+\infty} ds \tilde{\alpha}(s) u_s(x) = \frac{r(x)}{\sqrt{z'(x)}} \varphi(x) \\ &\simeq \sum_{l=1}^n \left[\sqrt{q} \frac{r(a_l)}{\sqrt{z'(a_l)}} \eta_{a_l}(x) + \sqrt{1 - q} \frac{r(b_l)}{\sqrt{z'(b_l)}} \eta_{b_l}(x) \right], \end{aligned} \quad (7.336)$$

where we have written

$$r(x) := \frac{(-1)^{l+1}}{\sqrt{2\pi}} \frac{P(x)}{\sqrt{-\prod_{j=1}^n (x - a_j)(x - b_j)}}, \quad (7.337)$$

and $\eta_c(x)$ is a wave packet sharply concentrated around $x = c$ and normalized according to $\int |\eta_c(x)|^2 dx = 1$.

⁷⁵In expression (7.335) we assume that $\int_{-\infty}^{+\infty} ds |\tilde{\alpha}(s, k)|^2 = 1$.

For example, if we choose conveniently the normalized eigenfunction given by

$$P_k^{(a)}(x) := \sqrt{\frac{\prod_{j=1}^n (b_j - a_k)}{\prod_{j \neq k} (a_j - a_k)}} \frac{\prod_{j \neq k} (x - a_j)}{\sqrt{-\prod_{j=1}^n (x - a_j)(x - b_j)}}, \quad (7.338)$$

we have that $\frac{r_k(a_l)}{\sqrt{z'(a_l)}} = \delta_{kl}$. Choosing $q_k = 1$, eq. (7.336) simplifies to

$$\alpha(x) \simeq \eta_{a_k}(x). \quad (7.339)$$

Similarly, we can choose the normalized eigenfunction given by⁷⁶

$$P_k^{(b)}(x) := \sqrt{\frac{\prod_j (b_k - a_j)}{\prod_{j \neq k} (b_k - b_j)}} \frac{\prod_{j \neq k} (x - b_j)}{\sqrt{-\prod_j (x - a_j)(x - b_j)}}. \quad (7.340)$$

In this case, we have that $\frac{r_k(b_l)}{\sqrt{z'(b_l)}} = \delta_{kl}$, and choosing $q_k = 0$, (7.336) simplifies to

$$\alpha(x) \simeq \eta_{b_k}(x). \quad (7.341)$$

In other words, in the limit of small modular parameter $s \simeq 0$, there is always a wave packet for the fermion intertwiner localized around any chosen endpoint. A general intertwiner can be constructed as a superposition of such endpoints localized wave packets

$$\alpha(x) \simeq \sum_{l=1}^n (e^{i\phi_{a_l}} \sqrt{p_{a_l}} \eta_{a_l}(x) + e^{i\phi_{b_l}} \sqrt{p_{b_l}} \eta_{b_l}(x)). \quad (7.342)$$

According to the normalization relation (7.298) for the wave packet and the localization properties of the functions $\eta_{a_l}(x)$ and $\eta_{b_l}(x)$, the probabilities p_{a_l} , p_{b_l} and the phases ϕ_{a_l} , ϕ_{b_l} can be freely chosen with the exception that they must satisfy

$$\sum_{l=1}^n p_{a_l} + \sum_{l=1}^n p_{b_l} = 1. \quad (7.343)$$

Thus, we can picture these wave functions as a quantum mechanical degree of freedom in a Hilbert space with one basis vector for each endpoint.

⁷⁶We remark that the eigenfunctions (7.338) and (7.340) are not orthogonal.

7.3.1.5. Free massive fermion in d dimensions and the Rindler wedge

We consider the right Rindler region $\mathcal{C} := \{\bar{x} \in \mathbb{R}^{d-1} : x^1 > 0\}$. In this case, by dimensional reduction, the eigenfunctions must be of the form

$$u_{s,k}(x^1, \bar{x}_{\parallel}) := w_s(\bar{k}, m, x^1 \sqrt{\bar{k}^2 + m^2}) e^{i\bar{k} \cdot \bar{x}_{\parallel}}, \tag{7.344}$$

where $\bar{x}_{\parallel} := (x^2, \dots, x^{d-1})$ are the transverse directions to the wedge, and $\bar{k} := (k^2, \dots, k^{d-1})$ are the momentum variables in that directions (degeneracy parameters for the modular eigenfunctions). Each of the components⁷⁷ $w_{s,j}$ ($j = 1, \dots, n$) of the spinor function, w_s satisfy the Euclidean Klein-Gordon equation $(-\nabla^2 + \sqrt{m^2 + \bar{k}^2}) w_{s,j} = 0$. Hence, $u_{s,k}$ behaves for the parallel direction x^1 as the massive 1 + 1 eigenfunction with reduced mass $\sqrt{m^2 + \bar{k}^2}$. The relative phases between each spinor components must to be adjusted in order to the full spinor solution $u_{s,k}$ satisfies the Euclidean Dirac equation. For any d dimensions, this is a cumbersome problem involving higher dimensional gamma matrices. In order to pleasantly display the main features of the problem, we study the case $d = 3$, where the spinor space is the same as in $d = 2$. Following the arguments above, we obtain the eigenfunctions

$$u_{s,k}(x^1, x^2) = \frac{1}{\pi} \frac{\sqrt{\cosh(\pi s)}}{\sqrt[4]{m^2 + k^2}} \begin{pmatrix} \sqrt{m^2 + k^2} K_{\frac{1}{2}-is}(x^1 \sqrt{m^2 + k^2}) \\ (k - im) K_{\frac{1}{2}+is}(x^1 \sqrt{m^2 + k^2}) \end{pmatrix} \frac{e^{ikx^2}}{\sqrt{2\pi}}. \tag{7.345}$$

To maximize the intertwiner correlation, we must choose a modular wave $\tilde{\alpha}(s, k)$ highly supported around $s \simeq 0$. This implies that the shape of $\tilde{\alpha}(s, k)$ in the k variable can be freely chosen. To better understand this, we can start with a wave packet of the form $\tilde{\alpha}(s, k) := \tilde{\alpha}_1(s) \tilde{\alpha}_2(k)$ and define

$$\beta(x^1, k) := \int_{-\infty}^{+\infty} ds \tilde{\alpha}_1(s) \frac{1}{\pi} \frac{\sqrt{\cosh(\pi s)}}{\sqrt[4]{m^2 + k^2}} \begin{pmatrix} \sqrt{m^2 + k^2} K_{\frac{1}{2}-is}(x^1 \sqrt{m^2 + k^2}) \\ (k - im) K_{\frac{1}{2}+is}(x^1 \sqrt{m^2 + k^2}) \end{pmatrix}, \tag{7.346}$$

which implies that

$$\alpha(x^1, x^2) = \int_{-\infty}^{+\infty} dk \tilde{\alpha}_2(k) \beta(x^1, k) \frac{e^{ikx^2}}{\sqrt{2\pi}}. \tag{7.347}$$

Now, we do a similar analysis as in the previous section. For any given fixed k , the function $\beta(x^1, k)$ is sharply localized around $x^1 \simeq 0$. Being more specific, given any $s_{max} \gtrsim 0$ and $\epsilon \gtrsim 0$, we can always choose a modular wave packet $\tilde{\alpha}_1(s)$ with support (essentially) contained in $[-s_{max}, s_{max}]$ such that the function $\beta(x^1, k = 0)$ is localized with probability almost 1 in the interval $[0, \frac{\epsilon}{m}]$.⁷⁸ Then, the supports of $\beta(x^1, k)$ for the other k modes,

⁷⁷For d dimensions, the spinor space has dimension $2^{\lfloor \frac{d}{2} \rfloor}$.

⁷⁸For example, as we have done in the previous section, we can take a Gaussian wave packet $\tilde{\alpha}(s)$ with dispersion $\sigma = s_{max} \gtrsim 0$ and mean value $|\lambda| \gg \frac{1}{\sigma}$. Then, $\epsilon = e^{-|\lambda| + \frac{1}{\sigma}}$.

will be (essentially) localized in the intervals $\left[0, \frac{\epsilon}{\sqrt{m^2+k^2}}\right] \subset \left[0, \frac{\epsilon}{m}\right]$, or in other words, the whole function $\beta(x^1, k)$ will be localized in the interval $x^1 \in \left[0, \frac{\epsilon}{m}\right]$ for all k . We can still freely choose the shape of the wave packet $\tilde{\alpha}_2(k)$ and get any shape to the wave packet $\alpha(x^1, x^2)$ in the transverse x^2 -direction, without affecting the relation $\langle V_1 V_2^\dagger \rangle \simeq 1$ for the intertwiner. In other words, the wave packet $\alpha(x^1, x^2)$ can be chosen such that

$$\langle V_1 V_2^\dagger \rangle \simeq 1 \Rightarrow \alpha(x^1, x^2) \simeq 0 \text{ for } x^1 > \frac{\epsilon}{m}, \quad (7.348)$$

and

$$\gamma(x^2) := \int_0^\infty dx^1 |\alpha(x^1, x^2)|^2 \text{ with any shape.} \quad (7.349)$$

The above discussion can be extended to any region $\mathcal{C} := A \times \mathbb{R}^{d-2}$ with $A \subset \mathbb{R}$. In particular, when $A := \bigcup_{l=1}^n (a_l, b_l)$ is any multi-interval, the fermion intertwiner $\alpha(x^1, \bar{x}_\parallel)$ wave function will behave as

$$\alpha(x^1, \bar{x}_\parallel) \simeq \sum_{l=1}^n f_{a_l}(\bar{x}_\parallel) \eta_{a_l}(x^1) + f_{b_l}(\bar{x}_\parallel) \eta_{b_l}(x^1), \quad (7.350)$$

where $\eta_c(x)$ is a wave packet highly localized around $x = c$ with $\int |\eta_c(x)|^2 dx = 1$, and $f_c(\bar{x}_\parallel)$ are arbitrary functions in the transverse directions satisfying

$$\sum_{l=1}^n \int_{\mathbb{R}^{d-2}} dx_\parallel \left(|f_{a_l}(\bar{x}_\parallel)|^2 + |f_{b_l}(\bar{x}_\parallel)|^2 \right) = 1. \quad (7.351)$$

7.3.1.6. Double cones in CFTs

Suppose we have a sphere in a CFT and we have an Abelian sector with unitary charge creating operator inside a double cone W_1 , which for simplicity we take $W_1 := D(\mathcal{C}_R)$. The operator has to be chosen such as to have almost zero modular energy. The double cone can be conformally mapped to a hyperbolic space, with curvature scale R and temperature $(2\pi R)^{-1}$ [35]. In this space, the modular Hamiltonian is just $K = 2\pi R H$, where H is the ordinary Hamiltonian in the hyperboloid. To produce a unitary operator V with small modular energy, the excitation has to carry low momentum. This requires it to be spread on regions much bigger than the curvature radius. On the other hand, it can be placed anywhere in the translational invariant hyperbolic space. However, once mapped back to the sphere, it will be highly concentrated along the boundary of the sphere in the Minkowski spacetime, but can be spread in the angular coordinates.

7.3.2. Free examples for finite groups

In this section, we study a simple example of intertwiner lower bounds for finite groups. Let us think we have independent free fermion fields $\psi_j(x)$ ($j = 1, \dots, n$) and consider symmetries that interchange the different fields. The field algebra \mathfrak{F} acts on the Hilbert space

$$\mathcal{H} := \bigotimes_{j=1}^n \mathcal{H}_j, \quad (7.352)$$

where all \mathcal{H}_j are equal to the Fock Hilbert space of a single fermionic operator. The vacuum vector is given by the tensor product of the vacuum vector in each factor, i.e. $|0\rangle := \bigotimes_{j=1}^n |0_j\rangle$. To define the action of the fermion field $\psi_j(x)$ in \mathcal{H} we have to invoke the \mathbb{Z}_2 -grading operators Γ_j which act as

$$\Gamma_j \bigotimes_{k=1}^n |v_k\rangle = (-1)^{|v_j|} \bigotimes_{k=1}^n |v_k\rangle, \quad (7.353)$$

where $|v_j|$ is the fermionic number of the vector $|v_j\rangle \in \mathcal{H}_j$. Then, the field operator $\psi_j(x)$ is represented in \mathcal{H} as

$$\psi_j(x) \mapsto \Gamma_1 \otimes \cdots \otimes \Gamma_{j-1} \otimes \psi_j(x) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \quad (7.354)$$

in order to have the right anticommutation relations, i.e. $\{\psi_j(x), \psi_k(y)\} = 0$ for all $j \neq k$ and all $x, y \in \mathbb{R}^d$. The local algebra of a bounded region $\mathcal{O} \in \mathcal{K}$ is defined as

$$\mathfrak{F}(\mathcal{O}) := \left\{ \sum_{j=1}^n \left[\psi_j(f_j) + \psi_j^\dagger(g_j) \right] : \text{supp}(f_j), \text{supp}(g_j) \subset \mathcal{O} \right\}'' , \quad (7.355)$$

and for an unbounded region $B \in \mathcal{K}$, $\mathfrak{F}(B)$ is defined imposing additivity (condition 2 in assumption 7.5). As in the case of a single fermion field, this net is “complete”, i.e. it satisfies the axioms 1, 2', and 3-5 of definition 2.75.

Here we consider two different symmetry groups: S_n the permutation group of n elements, and \mathbb{Z}_n the cyclic permutation group of n elements. These groups acts naturally on \mathfrak{F} by permuting the field operators $\psi_j(x)$ and giving place to two different observable algebras.⁷⁹

As in the previous section, we consider two nearly complementary regions W_1 and W_2 . We can build charge generating operators using the same type of operator V_j of equation (7.47) used in the preceding section. To simplify calculations, and since we are not interested in the fermion character of the fields but on the permutation symmetries

⁷⁹The observable algebra defined in this way contains operators which anticommute at spacelike distance. This issue is unimportant for the outcome of this section.

between different fields, we are going to use bosonic operators B_j for each field. These can be constructed out of the product of two of the V_j operators corresponding to non-overlapping test functions in the same region W_1 ,

$$B_j := iV_{j,x}V_{j,y}, \tag{7.356}$$

such that $B_j^2 = \mathbf{1}$ and $B_j^\dagger = B_j$. We also want these operators to have very small vacuum expectation value $\langle B_j \rangle \simeq 0$, what can be done by taking modes with small correlation.⁸⁰ These operators commute for different fields. As discussed in the previous section, we can choose $B_j^1 \in \mathfrak{F}(W_1)$ and $B_j^2 \in \mathfrak{F}(W_2)$ such that $\langle B_j^1 B_j^2 \rangle \simeq 1$. It is also clear that the expectation values $\langle B_j^1 B_k^2 \rangle = 0$ for any $j \neq k$.

To start, let us take $n = 3$ and the group $G := \mathbb{Z}_3$ of cyclic permutations of the fields ($|G| = 3$). Let us take the finite dimensional subalgebra \mathcal{B}_{W_1} generated by the unitaries B_1, B_2, B_3 of the three fields inside W_1 . These unitaries are chosen in such a way the group elements transforms one into the others. We have that $\dim(\mathcal{B}_{W_1}) = 2^3 = 8$. This algebra is generated by the projectors

$$P_{j,\pm} := \frac{\mathbf{1} \mp B_j}{2}, \quad P_{j,\pm}^2 = P_{j,\pm} = P_{j,\pm}^\dagger, \quad P_{j,+} P_{j,-} = 0, \tag{7.357}$$

or alternatively, it is also generated by the following set of eight orthogonal projectors

$$P_{\pm\pm\pm} := P_{1,\pm} P_{2,\pm} P_{3,\pm}. \tag{7.358}$$

We call simply P_β to these projectors, where $\beta = 1, \dots, 8$. There is an analogous algebra \mathcal{B}_{W_2} in region W_2 , and define $\mathcal{B}_{12} := \mathcal{B}_{W_1} \vee \mathcal{B}_{W_2}$. The vacuum state just gives non-zero expectation value to the same projector in W_1 and W_2 , and we have

$$\omega(P_\beta^1 P_{\beta'}^2) = \frac{1}{8} \delta_{\beta,\beta'}. \tag{7.359}$$

This state has vN entropy $S_{\mathcal{B}_{12}}(\omega) = \log(8)$.

Under the action of the group $G = \mathbb{Z}_3$, the above eight projectors in W_1 are interchanged in the following form. There are two regular representations (of three elements) spanned by the projectors with β having two plus signs or with two minus signs, and two trivial representations due to the projectors with all signs equal. Each representation matches with one corresponding representation in W_2 . All the projectors $P_\beta^1 P_{\beta'}^2$ of the same regular

⁸⁰We can take $V_{j,x}, V_{j,y}$ as in (7.47) with test functions $f_{j,x}, f_{j,y}$ having support inside small spheres of radius $R_{j,x}, R_{j,y}$ centered at spacelike separated points $x, y \in W_1$. Then, it is enough to take the $\text{dist}(x, y) \gg R_{j,x}, R_{j,y}$.

representation lead to the same expectation value

$$\omega \circ \varepsilon(P_\beta^1 P_{\beta'}^2) = \frac{1}{8} \times \frac{1}{3}, \quad (7.360)$$

because the conditional expectation mixes β, β' on all the possible values of the representation. Then, each regular representation contributes with $-9 \times 1/8 \times 1/3 \log(1/8 \cdot 1/3)$ to $S_{\mathcal{B}_{12}}(\omega \circ \varepsilon)$, while the trivial representations contribute with the same amount as for the entropy of ω , i.e. $-1/8 \log(1/8)$ each. Then, we get the lower the bound⁸¹

$$\Delta I \geq S_{\mathcal{B}_{12}}(\omega \circ \varepsilon) - S_{\mathcal{B}_{12}}(\omega) = \frac{3}{4} \log(3). \quad (7.361)$$

This coincides with the general result (7.129).⁸² In the factor $3/4$, we recognize the total probability of the regular representations, which equals $6/8$.

To improve this bound we add a new site on each region, i.e. we take two modes for each field and we call then B_j^α , where $\alpha = a, b$. Let us assume that the modes are decoupled, i.e. $\langle B_j^a B_j^b \rangle \simeq 0$. In other words, there is no entanglement between the two modes. We expect that this automatic holds if the modes commute with each other, e.g. if they are spatially separated, since by monogamy of entanglement, they cannot have correlations between them if they are maximally entangled with modes in the complementary region W_2 . Now, the algebra \mathcal{B}_{W_1} is spanned by the orthogonal projectors

$$P_\beta := P_{\pm\pm, \pm\pm, \pm\pm} := P_{1,\pm}^a P_{1,\pm}^b P_{2,\pm}^a P_{2,\pm}^b P_{3,\pm}^a P_{3,\pm}^b, \quad \beta = 1, \dots, 2^6 = 64. \quad (7.362)$$

Now, when we apply the group transformations we will have a larger proportion of regular representations because there are four possibilities ($\pm\pm$) for each field, and the three fields have to have equal this index in order not to form a regular representation. In general, taking N independent sites we get that the probability of the regular representation is $(1 - \frac{1}{2^{2N}})$, and following the same calculation as above we arrive at

$$\Delta I \geq \left(1 - \frac{1}{2^{2N}}\right) \log(3). \quad (7.363)$$

This means that our lower bound can approach $\log |G|$ as much as we want.

Different groups can be treated similarly. Let us take the non-Abelian group $G := S_3$ which has $|G| = 6$. Using N sites, we again get for each field 2^N labels for the projectors. Then, starting with one of the projectors, in order to the permutations of the fields do not generate $3! = 6$ different projectors (and hence the regular representation), it must be that

⁸¹Here we use that the conditional expectation preserves the trace, and hence $S_{\mathcal{B}_{12}}(\omega | \omega \circ \varepsilon) = -\Delta S_{\mathcal{B}_{12}}$.

⁸²In section 7.2.2, we got $S_{\mathcal{B}_{12}}(\omega) = 0$ because we choose a bigger non-commutative algebra containing the projectors to the diagonal elements we are using here. Even, if the two entropies change when we enlarge the algebra in this way, the RE given by the difference $S_{\mathcal{B}_{12}}(\omega \circ \varepsilon) - S_{\mathcal{B}_{12}}(\omega)$ does not change.

at least two of the labels for the different fields are equal. Then, the probability of the regular representation is the same as the probability of having the three labels different. As it was shown in section 7.2.2, the regular representation always contribute a $\log |G|$. This gives

$$\Delta I \geq \frac{(2^N - 1)(2^N - 2)}{2^{2N}} \log 6. \quad (7.364)$$

We need more an more sites for better precision, but the approach is exponentially fast.

It is evident that an example for the permutation group S_n can be constructed in the same way by using n fields. Since each finite group is a subgroup of a permutation group, and the regular representation of the permutation group decomposes into regular representations of the subgroups, an example can be devised in the same lines for any finite group.

7.3.3. Intertwiners at a finite distance

In subsection 7.3.1 we have seen that the intertwiners are concentrated around the boundary for complementary regions. Here we want to show that they will spread out in the coordinates orthogonal to the boundary if the two regions are separated. We cannot use now the vacuum modular conjugation to obtain a good charge creating operator partner. Then, we simply minimize the expectation value to obtain the optimal intertwiner.

Here we still deal with the simple case of the \mathbb{Z}_2 -symmetry of the free fermion. Then, we consider spacetime regions $W_j := D(\mathcal{C}_j)$, where $\mathcal{C}_1, \mathcal{C}_2 \subset \Sigma_0$ are spacelike regions over the Cauchy surface $\Sigma_0 := \{x^0 = 0\} \subset \mathbb{R}^d$. Moreover, $\bar{\mathcal{C}}_1 \subset \mathcal{C}'_2$ which means $W_1 \not\ll W_2$. We construct an intertwiner $\mathcal{I}_{12} := V_1 V_2^\dagger$ between W_1 and W_2 as in (7.296) and (7.297). The vacuum expectation of such a intertwiner is

$$\begin{aligned} \langle \mathcal{I} \rangle &= \int_{\mathcal{C}_1 \times \mathcal{C}_2} d^{d-1}x d^{d-1}y [\alpha_1(\bar{x})^\dagger C(\bar{x} - \bar{y}) \alpha_2(\bar{y}) - \alpha_2(\bar{x})^\dagger C(\bar{x} - \bar{y}) \alpha_1(\bar{y})] \\ &= \int_{\mathcal{C}_1 \times \mathcal{C}_2} d^{d-1}x d^{d-1}y \alpha_1(\bar{x})^T [C(\bar{x} - \bar{y}) - C(\bar{x} - \bar{y})^*] \alpha_2(\bar{y}), \end{aligned} \quad (7.365)$$

where $C(\bar{x} - \bar{y}) := \langle \psi(\bar{x}) \psi^\dagger(\bar{y}) \rangle$ is the fermion correlator, and we have used that the support of the two functions are disjoint. In the last equality in (7.365), we have assume that the smearing functions $\alpha_j(\bar{x})$ are real. When \mathcal{C}_1 and \mathcal{C}_2 are two spheres of size R separated by a distance $L \gg R$, we have that this expectation value falls as $(R/L)^{d-1}$ in the massless case and exponentially in the massive one.

Taking variations of (7.365) with respect to α_1 and α_2 with the constraints

$\int_{\mathcal{C}_j} \alpha_j(\bar{x})^T \alpha_j(\bar{x}) = 1$, we get

$$\int_{\mathcal{C}_1} d^{d-1}x \alpha_1^T(\bar{x}) [C(\bar{x} - \bar{y}) - C(\bar{x} - \bar{y})^*] = \lambda_2 \alpha_2(\bar{y}), \quad (7.366)$$

where λ_2 is a constant, the Lagrange multiplier. We have an analogous equation for α_1 . The solutions of these integral equations are generally not easy to obtain, but we can think for example in the easy case of very separated regions. In that case, the correlator function is almost constant when \bar{x}, \bar{y} belong to each of the regions. Then, it follows that the optimal distribution is given by constant functions α_1, α_2 . Hence, the charged modes have spread as much as possible. One can easily compute the contribution to the entropy of this intertwiner and check that it is less than the MI for the fermion at large distances, while it has the same falling rate L^{d-1} (in the massless case), where L is the distance between the regions.

As an example, for a chiral fermion, we can compute the RE for one intertwiner mode. Here we consider two intervals $A_j := (a_j, b_j)$ of length $R_j := b_j - a_j$, separated by a distance $L := a_2 - b_1$. In this case, we have a two dimensional Abelian algebra $\mathcal{B}_{12} \cong \mathbb{C} \oplus \mathbb{C}$ generated by the elements $\{\mathbf{1}, U\}$, and the unitary intertwiner U is defined as⁸³

$$U_{12} := i\mathcal{I}_{12} = iV_1V_2^\dagger = i \int_{A_1 \times A_2} dx dy \alpha_1(x) \alpha_2(y) [\psi(x) + \psi^\dagger(x)] [\psi(y) + \psi^\dagger(y)]. \quad (7.367)$$

For simplicity, we have used real functions $\alpha_j(x)$ with support in A_j . The vacuum expectation value of such operator is

$$\begin{aligned} \langle U_{12} \rangle_\omega &= i \int dx dy \alpha_1(x) \alpha_2(y) [C(x-y) - C(y-x)] \\ &\simeq -\frac{1}{\pi L} \left[\int_{A_1} dx \alpha_1(x) \right] \left[\int_{A_2} dy \alpha_2(y) \right]. \end{aligned} \quad (7.368)$$

In the above expression we have used $C(x-y) = \langle \psi(x) \psi^\dagger(y) \rangle = \delta(x-y) + \frac{i}{2\pi} \frac{1}{x-y} \simeq \frac{i}{2\pi L}$ when $L \gg R_1, R_2$. For constants functions $\alpha_j(x)$ normalized according to $\int_{A_j} dx \alpha_j(x)^2 = 1$, we have $\int_{A_j} dx \alpha_j(x) = \sqrt{R_j}$, and hence

$$\langle U_{12} \rangle_\omega = -\frac{\sqrt{R_1 R_2}}{\pi L}. \quad (7.369)$$

On the other hand, the expectation value on the state $\omega \circ \varepsilon_{12}$ is $\langle U_{12} \rangle_{\omega \circ \varepsilon_{12}} = 1$.

The classical probability distribution of any state ψ on the abelian algebra \mathcal{B}_{12} is (p_1, p_2) , where $p_1 - p_2 = \langle U_{12} \rangle_\psi$ and $p_1 + p_2 = \langle \mathbf{1} \rangle_\psi = 1$. Then, we have the following two probabilities

⁸³We introduce a i pre-factor in the definition (7.367) in order to U be a unitary operator.

distributions

$$\omega \rightarrow \left(\frac{1}{2} - \frac{\sqrt{R_1 R_2}}{2\pi L}, \frac{1}{2} + \frac{\sqrt{R_1 R_2}}{2\pi L} \right), \quad (7.370)$$

$$\omega \circ \varepsilon_{12} \rightarrow \left(\frac{1}{2}, \frac{1}{2} \right). \quad (7.371)$$

Then, the RE restricted to the algebra \mathcal{B}_{12} between such states is

$$S_{\mathcal{B}_{12}}(\omega | \omega \circ \varepsilon_{12}) = \frac{R_1 R_2}{2\pi^2 L^2}, \quad (7.372)$$

which is strictly smaller than the mutual information in the field algebra $I_{\mathfrak{F}}(A_1, A_2) = \frac{R_1 R_2}{6L^2}$ (see (7.290)). The model \mathcal{A} does not contain the fermion and its mutual information at large distances falls with a larger power than the one of the fermion.

We can speculate on two reasons why (7.372) does not coincide with the fermion MI at large distances. First, the two dimensional algebra \mathcal{B}_{12} may be too small, and second, our election of the intertwiner is not good enough (for example, we can still make different elections multiplying the intertwiner U_{12} by any unitaries in $\mathcal{A}(A_1 \vee A_2)$).

As a commentary to the previous calculations in section 7.3.2, if we choose the different charged modes on each region such that they are not independent to each other, we clearly get a less optimal result. In the limit when these modes are maximally entangled with the complementary region, this just means we have to take non-overlapping modes for the different sites. However, if the two regions W_1, W_2 do not touch each other, we cannot produce maximally entangled modes, and the charged modes will have a finite width in the direction perpendicular to the boundary. In this case, even if we have several sites on W_1 that are spatially separated, in general, the correlation of these modes will not vanish. To improve the result we need to diminish these correlations as much as possible since these correlations between charged modes in W_1 are not intertwiner correlations. This means the modes tend to repel each other in the direction parallel to the boundary in order to maximize the bound.

7.3.4. Sharp twists have Gaussian correlations with area law

In general it is difficult to obtain an exact explicit expression for the twist operators having all the desired properties listed in section 7.1.3. For example, let consider a $U(1)$ symmetry with a conserved current J_μ and regions $W_1 := D(\mathcal{C}_R)$ and $W_2 := D(\mathcal{C}'_{R+\epsilon})$ with $\epsilon > 0$. Then, we can construct twist operators τ_k for the region W_1 acting trivially along

W_2 as

$$\tau_k := e^{ikQ_1}, \quad k \in \mathbb{R}, \quad (7.373)$$

$$Q_1 := \int_{S^{d-2}} d\Omega \int_{\mathbb{R}_{\geq 0}} dr r^{d-2} \int_{\mathbb{R}} dt \alpha(t) \gamma(r) J_0(x), \quad (7.374)$$

where γ and α are any smooth smearing functions such that

$$\gamma(x) := \begin{cases} 0, & r \geq R + \frac{\epsilon}{2}, \\ 1, & r \leq R. \end{cases} \quad (7.375)$$

and $\alpha(t) = 0$ for $|t| \geq \epsilon/2$, while $\int_{\mathbb{R}} dt \alpha(t) = 1$. These twists form a one-parameter group of unitaries ($\tau_{k_1} \tau_{k_2} = \tau_{k_1+k_2}$). They act as conjugation on the operators of $\mathfrak{F}(W_1)$ in the same way as the global symmetry group, and act trivially on the elements of $\mathfrak{F}(W_2)$. However, τ_k is not 2π -periodic. To obtain this periodicity one should deform the twist inside the shell $S := (W_1 \vee W_2)$. This can be accomplished using the split property (see [154]), but the result would have a less transparent expression. For the $U(1)$ case, as we have discussed in subsection 7.2.5, it turns out that the expectation values will fall fast with $|k|$ for small $\epsilon \gtrsim 0$. Then, in the limit $\epsilon \rightarrow 0^+$, we do not expect difference in the leading divergent term in ϵ of the MI if we use the twist operators of expression (7.374) rather than any twist operators fulfilling the group composition law of $U(1)$.

Because of the CPT symmetry, expectation values of odd powers of Q_1 vanish. For computing $\langle Q_1^2 \rangle$ we use that, because of conservation, the correlation function of the currents writes

$$\langle J_\mu(x) J_\nu(0) \rangle = (g_{\mu\nu} \nabla^2 - \partial_\mu \partial_\nu) C(x). \quad (7.376)$$

For a general CFT, $C(x)$ is as in (7.266). Integrating by parts we get

$$\langle Q_1^2 \rangle = \int d^d x d^d x' \alpha(t) \alpha(t') \beta(r) \beta(r') C(x - x'), \quad (7.377)$$

where $\beta(r) := \gamma'(r)$ is a smooth function with support in the shell. Keeping one point fixed and moving the other on the shell, the result is seen to be proportional to the area times the remaining integral. Because the result is dimensionless, it is universally given, in a CFT, by

$$\langle Q_1^2 \rangle = c \frac{R^{d-2}}{\epsilon^{d-2}}, \quad (7.378)$$

where the dimensionless constant c depends on the precise shape of the smearing functions. If there are mass scales in the theory nothing changes for the leading term as far as $\epsilon \gtrsim 0$ is in the UV regime.

To compute $\langle Q_1^4 \rangle$ exactly we should know the four-point functions of the current and

these functions depend on the specific details of the theory. However, if we want to compute the leading term in $\epsilon \rightarrow 0^+$ we can argue as follows. Because of conservation and translation invariance, the four-point function of the charge density J_0 can be written as a combination of spatial derivatives of some functions H of the coordinate differences. The bulk integral can be then integrated out to get integrals on the shell. One way to convince oneself of this, is that each of the four operators Q_1 do not depend on the smearing inside the ball and the flux of the current can be written in a different Cauchy surface, as long as the shell part is not changed. Then, as above, in the thin shell, the leading contribution comes from points of coincidence of the correlator functions H . But the behavior of H at coincidence points can be read off from the points of coincidence of the correlators of J_μ , which satisfy clustering properties. Then, the leading term comes from two pairs of coincidence points, and for each coincidence points we have the same contribution as for the two-point function. There are also three and four-point coincidences, but these give subleading terms since we lose powers of the area. Since we have three possible pairings between the four points, the leading term should read

$$\langle Q_1^4 \rangle \simeq 3 c^2 \left(\frac{R^{d-2}}{\epsilon^{d-2}} \right)^2. \quad (7.379)$$

With the same reasoning, we see that, for the purpose of computing the leading term for small $\epsilon \gtrsim 0$ in $\langle Q_1^n \rangle$, we can assume that Q_1 is a free operator with Gaussian statistics and Wick's theorem holds. The same conclusion arises from thinking that the charge fluctuations are a sum over a large number of independent fluctuations along the surface, and then using the central limit theorem. We then arrive to a Gaussian distribution

$$\langle \tau_k \rangle = \langle e^{ikQ_1} \rangle \simeq e^{-k^2 \frac{\langle Q_1^2 \rangle}{2}}. \quad (7.380)$$

For small enough ϵ , only twist operators τ_k with small k have non-zero expectation values. As explained in section 7.2.5, this leads to

$$\Delta I \simeq \frac{1}{2} \log \langle Q_1^2 \rangle \simeq \frac{d-2}{2} \log(R/\epsilon). \quad (7.381)$$

More generally, we expect that in the $\epsilon \rightarrow 0^+$ limit, twists for any finite group symmetry that affects the UV fix point should also have an area law

$$\langle \tau \rangle \simeq e^{-c \frac{R^{d-2}}{\epsilon^{d-2}}}. \quad (7.382)$$

We can argue that this has to be the case in the following way. For a sharp twist operator, the charges that they measure are formed by the tensor product of a large number $\sim R^{d-2}/\epsilon^{d-2}$ of independent charge fluctuations (representations) along the surface. These

form a large representation of the group, and because of the arguments in section 7.2.2, this representation is mainly formed by copies of the regular representation except for a fraction of the Hilbert space that is exponentially small in the number of fused representations. The expectation value of the twist, for any element of the group except the identity, is zero for the regular representation. Then, we get the leading behavior (7.382).

A well-known example is to take a QFT and replicate it N times. We can then take the subalgebra of pointwise fixed elements under cyclic permutations of operators between copies. The twist operator for this symmetry is the Rényi twist operator [5], with expectation value for small ϵ

$$\langle \tau_n \rangle = \text{tr} \rho_1^n = e^{-(n-1)S_n(W_1)} \simeq e^{-c \frac{R^{d-2}}{\epsilon^{d-2}}}, \quad (7.383)$$

where S_n is the Rényi EE of the region on the original model. Thus, the expectation value of the twist is exponentially small with an area law for the exponent. This coincides with the area law for EE.⁸⁴

7.4. Conclusions of the chapter

In the context of QFT, the definition and computation of meaningful information theoretic quantities can become extremely complicated. The reason is simple. The most basic building block, the EE, is infinite and therefore ill-defined. To overcome this obstacle, two natural avenues have been pursued in the past. The first and most natural thing to do is to regularize the QFT with a lattice, which makes EE finite. The problem is that we should only trust aspects of such EE that do not depend on the regularization scheme. Unfortunately, in several examples, it turns out that to obtain the expected universal results one needs to fine-tune the UV lattice definition, for example, by ad hoc choices of boundary operators/algebras. The second and most rigorous avenue is to consider MI or related quantities, which can be considered either directly in the continuum QFT or as limits of lattice quantities [100]. The advantage of this approach is that it is, in principle, free from ambiguities, but the surprise is that in some cases it apparently turns out not to provide the expected universal results. The questions are thus clear: How do we extract the universal terms in the expansion of the EE correctly and unambiguously? What are the new physical features involved?

The main objective of this chapter has been to study these problems for the case of theories with global symmetries. These symmetries have the property that charged operators can be constructed locally. In the context of algebraic QFT, these charged

⁸⁴The results of our work, for this particular scenario, give $NI_{QFT} - I_{\text{Rényi orbifold}} = \log(N)$, when $\epsilon \rightarrow 0^+$.

superselection sectors are called DHR (because of Haag, Doplicher, and Roberts [45, 46, 49]).

The solution to the problem stated above starts with the key observation that theories with DHR sectors have certain ambiguities in the assignment of algebras to regions. These ambiguities have been known for a long time (see [2] for example), and we have described them in section 7.1. The main important message in this regard is that in theories with DHR sectors it is not possible to assign algebras to regions in a satisfactory way, where this means, a way satisfying the properties of isotony, duality, additivity and intersection.⁸⁵ More concretely, for global symmetries, there is a clash between duality and additivity for certain topologically non-trivial regions. In particular, for two disconnected regions, such as the ones used to define EE through MI, the additive algebra of regions W_1 and W_2 , defined as usual as $\mathcal{A}_{W_1} \vee \mathcal{A}_{W_2}$, is not equal to the commutant algebra of the complementary region. In fact, we have a violation of the duality property

$$\mathcal{A}_{W_1} \vee \mathcal{A}_{W_2} \subsetneq \mathcal{A}'_{(W_1 \vee W_2)'} . \quad (7.384)$$

The reason for such a proper inclusion is that one can find neutral operators \mathcal{I}_σ , which are called intertwiners for group theoretic reasons, which do not belong to the additive algebra $\mathcal{A}_{W_1} \vee \mathcal{A}_{W_2}$ but commute with the algebra of the complementary region $\mathcal{A}'_{(W_1 \vee W_2)'}$. Basically, for localized charge creating operators V_σ^j , transforming according certain representation σ with dimension d_σ ($j = 1, \dots, d_\sigma$) of the symmetry group, one can form the neutral operator

$$\mathcal{I}_\sigma := \sum_{j=1}^{d_\sigma} V_{W_1}^j V_{W_1}^{j\dagger} , \quad (7.385)$$

where the subscript indicates the localization properties of the operator. From this expression, it is transparent that $\mathcal{I}_\sigma \in \mathcal{A}'_{(W_1 \vee W_2)'}$, but $\mathcal{I}_\sigma \notin \mathcal{A}_{W_1} \vee \mathcal{A}_{W_2}$.

In turn, this is due to the existence of twist operators $\tau_{[g]}$, labeled by the conjugacy classes of the global group, which basically implement the symmetry transformation just in one of the connected components of $W_1 \vee W_2$, but they belong to the neutral algebra as well, even in the non-abelian case. These twists do not belong to the additive algebra of the complementary region $(W_1 \vee W_2)'$, but since it is a symmetry transformation, it commutes with all $\mathcal{A}_{W_1} \vee \mathcal{A}_{W_2}$, which is composed of products of neutral operators. Most importantly, as it has been described in this chapter, these twist operators do not commute with the intertwiners (7.385).

We want to remark that these observations, the appearance of these intertwiners and twists when considering topologically non-trivial regions, do not depend on the regulariza-

⁸⁵Duality for two intervals in CFT in $d = 2$ is related to modular invariance. Then, duality in higher dimensions and different regions can also be thought as requirements generalizing the ones of modular invariance for $d = 2$ to other QFT and dimensions.

tion scheme. In particular, it does not depend on algebra choices in a lattice regularization. It is a true physical feature of the continuum QFT, a macroscopic manifestation of the underlying global symmetry group. It is also important to notice that these observations are purely made within the vacuum sector of the theory, no charge creating operator is needed, since both \mathcal{I}_σ and $\tau_{[g]}$ are neutral operators that indeed belong to the additive algebra of a sufficiently big spacetime region.

The solution to the problem stated above is rooted in the implications of the existence of such operators for the MI. From both, a technical and physical perspective, the section 7.2 has been devoted to analyzing the modifications to the MI due to this enlarged operator algebras. The main tool that has been used is the key formula (3.65) described in section 3.4. When applied to QFT for the inclusion of algebras $\mathcal{A} \subset \mathfrak{F}$ and choosing the conditional expectation and the states appropriately, (3.65) gives the difference of the MI on the two models between the regions W_1 and W_2

$$I_{\mathfrak{F}}(W_1, W_2) - I_{\mathcal{A}}(W_1, W_2) = S_{\mathfrak{F}}(\omega \mid \omega \circ \varepsilon_{12}), \quad (7.386)$$

implying that such a RE difference can be computed solely on the neutral algebra in the vacuum sector of the theory. Even more, $I_{\mathfrak{F}}$ has a natural and direct definition in \mathcal{A} (see (7.103)).

The fact that the difference of MIs is itself a RE greatly simplifies the analysis of such an object since one can resort to monotonicity and convexity to constraint it in several ways. More concretely, we have found two dual ways to study the problem. In the first approach, we compute a lower bound to such a RE by restricting to a certain finite algebra of intertwiners, constructed basically from (7.119). The challenge is to find the best finite intertwiner subalgebra, i.e a finite subalgebra providing the best lower bound to the RE. Interestingly, this maximization procedure requires two concrete physical ingredients. First, from a group theory point of view, we need to choose the intertwiner subalgebra associated with the regular representation of the group. Second, from the point of view of QFT, once such a regular representation is chosen, we have to make sure that we maximize the correlation functions in the vacuum state. This forces us to choose the intertwiners so as to commute as much as possible with the vacuum modular Hamiltonian. Explicit examples of this maximization of correlation functions and of how the regular representation is inherently present in the vacuum, have been described in section 7.3. The identification of these two physical features, the regular representation and choosing intertwiners that commute with the modular Hamiltonian, are two of the most important physical messages of our analysis.

The second line of study uses the equality of entanglement entropies for complementary algebras to relate the previous RE to another relative entropy on the complementary

algebra, which includes the additive algebra and the twists operators $\tau_{[g]}$. From this perspective, the problem is similar: we need to find the best subalgebra that provides the best upper bound. The connection with the intertwiner version is rooted in the fact that the group algebra has the same dimension as the regular representation. While the intertwiners are labeled by the irreducible representations of the group, the invariant twists can be labeled by conjugacy classes, and both labels run over the same number of elements.

Moreover, the twist/intertwiner duality is best described by both, the entropic certainty and uncertainty relations, which were derived in section 7.2.4 and nicely codify the non-commuting character of the twist/intertwiner algebras in an information theoretic manner. These uncertainty relations are also among the most important physical outcomes of our analysis.

Using these features, we have been able to compute the modifications to universal contributions to the MI associated with finite and continuous Lie groups. We have also computed the universal contributions when considering different topologies, excited states, scenarios with spontaneous symmetry breaking, and analyze the particularities of two dimensional QFTs. All these results have been described in section 7.2. Some of these results were found previously in the literature, and some of them are new. But we want to stress that all of them arise from the same basic physical principles discussed above. Hence, in this sense, the present approach provides a unification of all these seemingly disconnected results.

We want to end with some important remarks. It is sometimes said that the problems we have been considering in the present work arise in theories with gauge symmetries, and are due to a certain arbitrariness in the choice of algebras in lattice regularizations. Our first important remark is that this is wrong. The problems only appear when the operator algebra considered is incomplete and the theory has a structure of SS. To sense the difference, we could have a “gauge” theory with charges in all representations. This last theory has no problems of assignments of algebras to regions in any meaningful sense, where meaning is always related to properties of the continuum QFT. Indeed, the converse is also true, we can have theories with no gauge symmetry which actually show macroscopic ambiguities in the definition of the MI. All the cases considered in this paper are examples of such a scenario. The second important remark is that whenever we have a structure of SS, their contribution to the MI can be obtained only by focusing on the vacuum sector. This is pretty impressive and it can be related to the fact that the neutral algebra is an example of a sufficient algebra, whenever the state considered is invariant under the symmetry (see [64, 65] for the definition of a sufficient algebra).

Chapter 8

Conclusions

This thesis is advocated to the study of aspects of EE in QFT. We followed an algebraic perspective. As we argued in the introduction and chapters 2 and 3, the algebraic approach to QFT fits naturally with this purpose. On one hand, the local degrees of freedom of QFT organize themselves in algebras (subalgebras of the global algebra more precisely), leading to a map

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \tag{8.1}$$

from the set of spacetime regions to the set of subalgebras of the global algebra of QFT [2, 57, 59]. On the other hand, the natural way to define statistical/information measures is in the context of vN algebras. In fact, any vN algebra represents a statistical “quantum” system. These are the mathematical non-commutative generalizations of classical probability spaces, in the sense of Kolmogorov. In the later case, the measurable subsets (usually taken as the Borel subsets) of the probability space form a distributive lattice, giving place to usual classical rules between probabilities called *classical logic*. In the non-commutative case, the set of projectors of a vN algebra (which by the way generate all the algebra itself) form an orthomodular, but not distributive, lattice. This reflects the fact that there are non-commuting projectors, or equivalently, that the system has incompatible observables. Physical experiments involving systems described under this approach display a richer structure of outcomes called *quantum logic* [176–178].

This richer structure is still more significant when one considers bipartite quantum systems. Some states on such systems show a particular kind of correlations, between the subsystems, which are present only due to the quantum nature of the systems involved. This pure quantum phenomenon, known as entanglement, has already brought out several problems to philosophers of quantum theory, giving place to a collection of anti-intuitive gedanken and real experiments. Moreover, when used in our favor, it allows to construct skillful quantum devices (simulators or computers) to solve problems more efficiently than

devices that are governed by laws of classical physics.

In QFT, this phenomenon is present on fundamental grounds. The Reeh-Schlieder (RS) theorem asserts that the vacuum state, of any QFT, is entangled at any distance. The key point that makes this theorem true is the relativistic covariance of the theory and the spectrum condition. However, the RS theorem does not hold for general non-relativistic QFTs. This suggests a deep connection between geometry and entanglement. This connection has been subsequently enhanced in the context of quantum gravity due to the interpretation of the entropy of a black hole as EE, and its generalization with the holographic prescription for EE in AdS/CFT. From the pure QFT perspective, the RS theorem allows us to define, for any region of the spacetime, a state-dependent dynamics which leaves invariant the observables of the corresponding region. This evolution is called modular flow and is represented by a one-parameter group of unitaries whose Hermitian generator is called modular Hamiltonian. The modular Hamiltonian appears in many conceptual aspects of the foundations of AQFT. In particular, from our perspective, we remark the following two. On one hand, it is used to define statistical/entanglement measures for general quantum systems, e.g. the relative entropy (RE). RE is a central concept in the study of entanglement in QFT, because, in contrast to vN entropy, it is a well-defined quantity in the continuum QFT and it also allows us to derive many other quantum information measures, such as the mutual information (MI). On the other hand, it is deeply related to spacetime symmetries according to Bisognano-Wichmann (BW) theorem. In fact, this relation is so strong that, under general assumptions, we have that the upshot of BW theorem is equivalent to Poincaré covariance of the theory [95].

All these insights point out the relevance of the entanglement in QFT, and the benefits of the algebraic approach to study its consequences. Since AQFT is a perfect framework to study QFT rigorously, we proposed to compute entanglement measures and modular Hamiltonians from this perspective. The advantage of this approach is that it allows us to compute quantities, from precise mathematical grounds, in an unambiguous way, without the need to deal with cutoffs and cumbersome regularization prescriptions. In this way, we can argue that the relation between EE and AQFT is very strong. On one hand, EE entropy, or any entanglement measure, needs the algebraic approach to be properly well-defined. On the other hand, we may entertain the expectation that QFT may admit a complete precise formulation in terms of entropic quantities. In the Wightman approach to QFT, the knowledge of all vacuum expectation values of product of field operators (Wightman functions) is enough to reconstruct uniquely the theory. In the algebraic framework, the local algebras are the central objects in the description of the theory, and in this setting, the equivalent to the vacuum correlators are the statistical/entanglement measures attached to such algebras. In particular, the mutual information of the vacuum state universally quantifies such correlations for commuting algebras attached to spacelike separated

regions. We can expect that the knowledge of all the mutual informations for the vacuum state and any pair of regions, would be enough to reconstruct uniquely the AQFT model. One can also hope that it would help us in the search of a “dynamical principle” which allows us to construct non-trivial models satisfying the axioms of AQFT discussed along this thesis.

With this perspective in mind, in chapter 5, we computed the relative entropy for coherent states using Araki formula. The advantage of using this formula is that we could unambiguously compute the RE in the continuum QFT, without the need of invoking non-rigorous methods such as lattice or replica trick. We showed that using such non-rigorous methods, ambiguities can arise in the final result of such RE. These ambiguities come when one uses the formula

$$S(\phi | \omega) = \Delta\langle K \rangle - \Delta S, \quad (8.2)$$

which holds for finite quantum systems, but not for the local algebras of QFT. More precisely, both terms on the r.h.s. of (8.2) are mathematically ill-defined. The second one is ill-defined because the EE entropy is UV-divergent in QFT, and despite the difference of two EE could be finite, its value, in general, depends on the regularization prescription. The first term is trickier. It involves the “half” modular Hamiltonian, which, despite not being a mathematical well-defined operator, is a sesquilinear form, and hence, expectation values of this operator are generally finite. However, to define this operator, we need to cut the “full” modular Hamiltonian into two pieces, and in this act, ambiguities may appear in the boundary of the region. In the case of the free scalar field and the Rindler wedge, these ambiguities come from possible improving terms of the stress-tensor. Using the Araki formula, we solved these ambiguities in the case of coherent states. Moreover, the general structure of modular theory allowed us to prove general statements about the RE without the need of specifying any particular model. More concretely, we showed that, for free theories, the RE between coherent states is symmetric.

Following in the same direction, in chapter 6, we computed the modular Hamiltonian and the MI for a free chiral scalar field. A similar computation has been done in the past for the free chiral fermion [36]. We rederives such a result, in a more transparent way, using a novel method that relates the eigenfunctions of the correlator kernel with solutions of the wave equation in the Euclidean plane [38]. In this case, since the theory is massless, the wave equation is equivalent to finding holomorphic functions in the complex plane with suitable boundary conditions, which is nothing else than a Riemann-Hilbert problem. For the free chiral fermion, the modular Hamiltonian is non-local whenever the regions contain more than one interval. However, in that case, the non-locality has a very particular structure: the non-local terms are quadratic in the fermion operator pairing each point in any given interval, with exactly one point in all others. The reason for this special structure could be, perhaps, the multi-local symmetries described by Rehren and

Tedesco [138]. Using the same techniques, we computed the modular Hamiltonian and the MI for the free chiral scalar for two intervals. However, in contrast to the fermion field, the non-local part is given by a smooth kernel, which convolutes any point in any of the two intervals with all the other points in both intervals. In both cases, the local part is given by an integral of the stress tensor smeared by a *local temperature* function. This function is universal for free fields [38], and we expect that it should be universal for any $d = 2$ CFT. The local term gives the leading contribution to the RE between the vacuum state and a highly-energetic and well-localized excitation inside the region [38]. Remarkably, the MI for the scalar is upper bounded by the one of the fermion field. This should not be surprising since the free chiral scalar net is a subnet of the free chiral fermion net. To our surprise, the scalar shows a failure of the duality condition (assumption 4 in definition 2.75) for the algebras assigned to two intervals. This fact is translated into a loss of a symmetry property for the mutual information usually associated with modular invariance [126]. In the analysis of this curious phenomenon, we identified which operators are the responsible for the failure of the duality, and how it can be restored if we merge the two chiralities in a $d = 2$ CFT in Minkowski spacetime. On the other hand, the free fermion satisfies the duality conditions and its MI satisfies the mentioned symmetry property.

This last observation motivated the study of our last chapter 7. This led us to investigate the origin and scope, among general QFTs in any dimensions, of the failure of the duality behind the failure of the symmetry property in the EE. There we studied entanglement aspects of theories having non-trivial superselection sectors (SS) on the EE entropy. The structure of SS has direct consequences in the considered model, which is accessible from the vacuum representation itself. In fact, it leaves a definite imprint in the relations between the different local subalgebras of operators assigned to the different regions of the spacetime. The superselection structure affects the relations between algebras and regions, either violating the property of duality and/or additivity for some topologically non-trivial regions. In this new scenario, where models with SS sectors are considered, there is more than one choice for the algebra of topologically non-trivial regions, being this directly sensitive to the EE. We constructed an entropic order parameter given by the difference of two MIs between two disconnected regions. These MIs can be reinterpreted as corresponding to different models, with and without SS. We found that such a difference is bounded above by the dimension of the global symmetry group and saturates whenever the regions touch each other. Moreover, such an entropic parameter leads to a certainty entropic relation involving REs in two non-commutative algebras. The non-commutativity of such algebras is due to the operators that break the duality condition for non-connected regions, and are directly related to the SS of the model. Furthermore, we argued that such a certainty relation has a very general structure that should hold for any inclusion of algebras and relates entropic quantities with the index theory of vN subfactors. The subtle certainty relation is

stronger than usual uncertainty relations between non-commuting operators. There is no prior example of this relation in the literature. We hope it should have wider applications than the ones discussed in this thesis. Once the entropic order parameter was recognized and fully characterized, we proceeded to apply it to different situations: continuous Lie symmetry groups, regions with other topologies, excited states, and scenarios with spontaneous symmetry breaking. We successfully obtained the relevant universal contribution in all of these scenarios, unifying, generalizing, and clarifying some disperse specific results that have appeared in the literature, as well as, obtaining some previously unknown ones.

We expect that the contributions we have made in this field will be useful to better understand the structure of entanglement in QFT. In particular, with the motivation of finding a possible quantum information version of QFT, we think that we have made great progress in understanding the connection of EE with internal symmetries, which are very important in particle physics. On the other hand, this connection could be extended to more general scenarios. For example, in the case of gauge symmetries, the mismatch between the A-anomaly coefficient and the coefficient of the logarithmic term in the EE of a Maxwell field could be solved if we compute the MI for the complete model that includes the charged fields, instead of the “incomplete” model of the pure gauge Maxwell field [179]. In fact, the difference in the mismatch should be computable using the techniques and the entropic order parameter we have developed along chapter 7.¹ Another important application of the results of that chapter is in the context of AdS/CFT. In [180], we also argue that the features of holographic EE correspond to a picture of a sub-theory with a large number of superselection sectors.² In fact, in the holographic context, the formula (7.101) could be interpreted as the Ryu-Takayanagi formula plus quantum corrections. In that case, when it is applied to connected regions on the boundary theory, it gives the known result of “relative entropy equals bulk relative entropy” [10]. This interpretation of the features of holographic EE in terms of sub-theory and conditional expectations would open the way to interpret the bit-thread picture of holographic entanglement in terms of more concrete and physical objects, the intertwiners, in the boundary theory.

¹Though this investigation is well-advanced and shortly to be published, it is not included in this thesis.

²This investigation is more speculative to fit in the design of this thesis, and for this reason, has not been included in the present volume.

Appendix A

Lattices

Definition A.1. Let (L, \leq) be a partially ordered set and $S \subset L$ a subset. An element $u \in L$ is said to be an *upper bound* of S if $s \leq u$ for all $s \in S$, and an element $l \in L$ is said to be a *lower bound* of S if $l \leq s$ for all $s \in S$.

Remark A.2. A set $S \subset L$ may have many upper bounds (or lower bounds), or none at all.

Definition A.3. Let (L, \leq) be a partially ordered set, and $S \subset L$ an arbitrary subset. An upper bound u of S is said to be a *supremum* if $u \leq \tilde{u}$ for all upper bounds \tilde{u} of S . Dually, a lower bound l of S is said to be an *infimum* if $\tilde{l} \leq l$ for all lower bounds \tilde{l} of S .

Remark A.4. A set $S \subset L$ may have many suprema (or infima), or none at all.

Definition A.5. A *lattice* is a partially ordered set (L, \leq) such that any pair of elements $\{a, b\} \subset L$ has a unique supremum, which is denoted by $a \vee b$, and a unique infimum, which is denoted by $a \wedge b$.

Definition A.6. A lattice L is called *bounded* if there exists a *greatest element* $1 \in L$ such that $l \leq 1$ for all $l \in L$, and a *least element* $0 \in L$ such that $0 \leq l$ for all $l \in L$.

Definition A.7. A bounded lattice L is called *complemented* if every element $a \in L$ has a *complement*, i.e. an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$.

Remark A.8. An element $a \in L$ of a complemented lattice may have many complements. A bounded lattice in which every element has exactly one complement is called a *uniquely complemented* lattice.

Definition A.9. A *complementation* on a complemented lattice L is a function that maps each element $a \in L$ to a complement $a' \in L$ in such a way that the following axioms are satisfied,

1. (complement laws) $a \wedge a' = 0$ and $a \vee a' = 1$,

2. (involution) $(a')' = a$,
3. (order-reversing) $a \leq b \Rightarrow b' \leq a'$,

for all $a, b \in L$. A complemented lattice equipped with a complementation is called an *orthocomplemented lattice*.

Example A.10. The prototypical example of an orthocomplemented lattice is the power set of any given set. In this case, the order is given by inclusion, the supremum (resp. infimum) of any two sets is the union (resp. the intersection) of such sets, and the complementation is given by the complement of sets.

Lemma A.11. *Let L be an orthocomplemented lattice. Then, we have that*

$$(a \vee b)' = a' \wedge b' \quad \text{and} \quad (a \wedge b)' = a' \vee b', \quad (\text{A.1})$$

for all $a, b \in L$. The relations (A.1) are known as *De Morgan laws*.

Definition A.12. Let (L, \leq) be a lattice and $S \subset L$ a subset. We say that S is a *sublattice* if $a \vee b \in S$ and $a \wedge b \in S$ whenever $a, b \in S$. In the case of orthocomplemented lattices, we also require that $a' \in S$ whenever $a \in S$. Automatically, (S, \leq) is an (orthocomplemented) lattice.

Definition A.13. A lattice L is said to be *distributive* if it satisfies any of the following two equivalent properties.

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for all $a, b, c \in L$.
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all $a, b, c \in L$.

Both relations above are usually called *distributive laws*.

Proposition A.14. *Any complemented and distributive lattice is uniquely complemented.*

Definition A.15. A distributive and orthocomplemented lattice is called *Boolean lattice*.

Definition A.16. Let L be an orthocomplemented lattice and $a, b \in L$. We say that a *commutes* with b if $a = (a \wedge b) \vee (a \wedge b')$. It can be shown that a commutes with b if and only if b commutes with a . In other words, commutation is a symmetric relation.

Proposition A.17. *Let L be an orthocomplemented lattice. Then, L is distributive (and hence Boolean) if and only if a commutes with b for all $a, b \in L$.*

Definition A.18. An orthocomplemented lattice L is said to be *orthomodular* if

$$a = (a \wedge b') \vee b, \quad \text{for all } b \leq a. \quad (\text{A.2})$$

Remark A.19. Any Boolean lattice is orthomodular, but the converse is false.

Definition A.20. A *lattice homomorphism* is a map $\Phi : L_1 \rightarrow L_2$ between lattices L_1, L_2 , such that Φ preserves all the lattice operations, i.e.

$$\Phi(a \vee b) = \Phi(a) \vee \Phi(b) \quad \text{and} \quad \Phi(a \wedge b) = \Phi(a) \wedge \Phi(b), \quad (\text{A.3})$$

for all $a, b \in L_1$. In the case of orthocomplemented lattices, we also require that $\Phi(a') = \Phi(a)'$ for all $a \in L_1$.

Definition A.21. A lattice homomorphism $\Phi : L_1 \rightarrow L_2$ is a *lattice isomorphism* if Φ is invertible and $\Phi^{-1} : L_2 \rightarrow L_1$ is also a lattice homomorphism. When $L_1 = L_2 = L$, any lattice isomorphism $\Phi : L \rightarrow L$ is called a *lattice automorphism*. The set of all lattice automorphisms is denoted by $\text{Aut}(L)$ and it forms a group under composition.

Appendix B

Sobolev spaces

For the definition and properties of Sobolev spaces, we follow [181]. However, here we adapt the notation to our convenience.

Consider the test function space $\mathcal{D}(\mathbb{R}^n) := C_c^\infty(\mathbb{R}^n) \subsetneq \mathcal{S}(\mathbb{R}^n)$ of smooth and compactly supported functions with its usual topology. The n -dimensional complex *Sobolev space* of order $\alpha \in \mathbb{R}$ is defined as

$$H^\alpha(\mathbb{R}^n) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : \hat{f}(\bar{p}) \omega_{\bar{p}}^\alpha \in L^2(\mathbb{R}^n) \right\}, \quad (\text{B.1})$$

where $\omega_{\bar{p}} = \sqrt{\bar{p}^2 + 1}$ and $\hat{f}(\bar{p}) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\bar{x}) e^{-i\bar{p}\cdot\bar{x}} d^n x$ is the usual Fourier transform. From the definition follows that $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and $H^\alpha(\mathbb{R}^n) \subset H^{\alpha'}(\mathbb{R}^n)$ if $\alpha > \alpha'$.

The Sobolev space $H^\alpha(\mathbb{R}^n)$ is a Hilbert space under the inner product

$$\langle f, g \rangle_{H^\alpha} := \langle \hat{f} \omega_{\bar{p}}^\alpha \mid \hat{g} \omega_{\bar{p}}^\alpha \rangle_{L^2} = \int_{\mathbb{R}^n} d^n p \hat{f}(\bar{p})^* \hat{g}(\bar{p}) \omega_{\bar{p}}^{2\alpha}. \quad (\text{B.2})$$

Furthermore, for $f \in H^\alpha(\mathbb{R}^n)$ we have that $\|f\|_{H^{\alpha'}} \leq \|f\|_{H^\alpha}$ if $\alpha > \alpha'$, and hence the natural injections $H^\alpha(\mathbb{R}^n) \hookrightarrow H^{\alpha'}(\mathbb{R}^n)$ for $\alpha > \alpha'$ are continuous. We also have that the set $C^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ is dense in $H^\alpha(\mathbb{R}^n)$.

When $\alpha = k \in \mathbb{N}_0$, there is also another useful equivalent characterization of the Sobolev spaces in term of weak derivatives¹

$$H^k(\mathbb{R}^n) = \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : D^\mu f \in L^2(\mathbb{R}^n), \text{ for all } |\mu| \leq k \right\}. \quad (\text{B.3})$$

It is useful to introduce a new norm in $H^k(\mathbb{R}^n)$ as

$$\|f\|'_{H^k} := \left(\sum_{|\mu| \leq k} \int_{\mathbb{R}^n} d^n x |D^\mu f(x)|^2 \right)^{\frac{1}{2}}, \quad (\text{B.4})$$

¹The weak derivative of an element of $\mathcal{D}'(\mathbb{R}^n)$ is its usual derivative in the distributional sense.

which is equivalent to the former norm $\|\cdot\|_{H^k}$.

The real Sobolev spaces $H^\alpha(\mathbb{R}^n, \mathbb{R})$ are defined in a similar manner as above, but restricting to real valued functions.

In general, it is easier to calculate the usual pointwise derivatives rather than the weak derivatives. Then, the following lemma states sufficient conditions for both notions of derivatives coincide. Before we formulate it, we need to introduce the notions of C^k -piecewise function.

Definition B.1. Let $U \subset \mathbb{R}^n$ open, $f \in L^1_{loc}(U)$ and $k \in \mathbb{N}_0$. We say that f is a C^k -piecewise function iff there exists a finite family of pairwise disjoint open sets $\{\Omega_j\}_{j=1, \dots, J} \subset U$ such that

1. $\bigcup_{j=1}^J \bar{\Omega}_j = \bar{U}$.
2. $f \in C^k(\Omega_j)$ for all $j = 1, \dots, J$.
3. For all $j = 1, \dots, J$, $\forall x_0 \in \partial\Omega_j$ and for all multi-index $|\alpha| \leq k$, the $\lim_{x \rightarrow x_0} D^\alpha f(x)|_{\Omega_j}$ exist and are finite, where D^α is the usual multi-order pointwise derivative.

We denote $C^k_t(U)$ the set of C^k -piecewise functions on U .

Now, we formulate the lemma that ensures that weak and pointwise derivatives coincide.

Lemma B.2. Let $U \subset \mathbb{R}^n$ be open and $f \in C^0(U) \cap C^1_t(U)$. Then, the (first order) weak derivatives of f coincides with the usual pointwise derivatives.

Proof. Since $f \in C^0(U) \cap C^1_t(U)$ we have that f is locally Lipschitz continuous on U (see Corollary 4.1.1 on [182]). Then, we have that f is locally absolute continuous on U , and hence, $f \in L^1_{loc}(U)$. Then f is weakly differentiable and the (first order) weak and pointwise derivatives of f coincide a.e. \square

Now, using the above lemma and the alternative definition (eq. (B.3)) for the Sobolev space $H^1(\mathbb{R}^n)$, the proof of lemma 5.11 is trivial.

Appendix C

Computations of chapter 5

C.1. Calculation of $\langle f_R | k_1 | f_R \rangle_{1p}$

Take $f_R \in \mathcal{S}(\mathbb{R}^d, \mathbb{R})$ and for simplicity call $f := f_R$. Then,

$$\begin{aligned} \langle f | k_1 | f \rangle_{1p} &= \operatorname{Re} \langle f | k_1 | f \rangle_{1p} = \operatorname{Re} \left(-i \frac{d}{ds} \Big|_{s=0} \langle f | e^{ik_1 s} | f \rangle_{1p} \right) = \\ &= \frac{d}{ds} \Big|_{s=0} \operatorname{Im} \langle f | u(\Lambda_1^s, 0) | f \rangle_{1p} = \frac{d}{ds} \Big|_{s=0} \operatorname{Im} \langle f | f^s \rangle_{1p}, \end{aligned} \quad (\text{C.1})$$

where we have defined $f^s := f_{(\Lambda_1^s, 0)}$. As we have explained in section 5.3.3, there exist functions $f_\varphi, f_\pi, f_\varphi^s, f_\pi^s \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ such that $E(f) = E_\varphi(f_\varphi) + E_\pi(f_\pi)$ and $E(f^s) = E_\varphi(f_\varphi^s) + E_\pi(f_\pi^s)$. Replacing these into (C.1), we get

$$\begin{aligned} \langle f | k_1 | f \rangle_{1p} &= \frac{d}{ds} \Big|_{s=0} \operatorname{Im} \langle f_\varphi + f_\pi | f_\varphi^s + f_\pi^s \rangle_{1p} = \frac{d}{ds} \Big|_{s=0} (\operatorname{Im} \langle f_\varphi | f_\pi^s \rangle_{1p} + \operatorname{Im} \langle f_\pi | f_\varphi^s \rangle_{1p}) \\ &= \frac{d}{ds} \Big|_{s=0} \left(\frac{1}{2} \int_{\mathbb{R}^{d-1}} f_\varphi(\bar{x}) f_\pi^s(\bar{x}) d^{d-1}x - \frac{1}{2} \int_{\mathbb{R}^{d-1}} f_\varphi^s(\bar{x}) f_\pi(\bar{x}) d^{d-1}x \right), \end{aligned} \quad (\text{C.2})$$

where we have used the relations (5.66) in the second line and (5.91) in the last line. From the Poincaré invariance of the distribution $\Delta(x)$, we have that

$$F^s(x) = \int_{\mathbb{R}^d} \Delta(x-y) f^s(x) d^d y, \quad (\text{C.3})$$

where we have defined $F^s := F_{(\Lambda_1^s, 0)}$. Then, according to (5.87), the following relations hold

$$f_\varphi(\bar{x}) = -\frac{\partial F}{\partial x^0}(0, \bar{x}), \quad (\text{C.4})$$

$$f_\pi(\bar{x}) = F(0, \bar{x}), \quad (\text{C.5})$$

$$f_\varphi^s(\bar{x}) = -\cosh(s) \frac{\partial F}{\partial x^0}(\bar{x}^s) + \sinh(s) \frac{\partial F}{\partial x^1}(\bar{x}^s), \quad (\text{C.6})$$

$$f_\pi^s(\bar{x}) = F(\bar{x}^s), \quad (\text{C.7})$$

where $\bar{x}^s := (-x^1 \sinh(s), x^1 \cosh(s), x_\perp)$. And hence,

$$\frac{d}{ds} \Big|_{s=0} f_\varphi^s(\bar{x}) = x^1 \frac{\partial^2 F}{(\partial x^0)^2}(0, \bar{x}) + \frac{\partial F}{\partial x^1}(0, \bar{x}), \quad (\text{C.8})$$

$$\frac{d}{ds} \Big|_{s=0} f_\pi^s(\bar{x}) = -x^1 \frac{\partial F}{\partial x^0}(0, \bar{x}). \quad (\text{C.9})$$

Replacing such expressions into (C.2), using the equation of motion for F and performing an integration by parts, we finally get

$$\langle f | k_1 | f \rangle_{1\text{p}} = \int_{\mathbb{R}^{d-1}} d^{d-1}x x^1 \frac{1}{2} \left(\left(\frac{\partial F}{\partial x^0} \right)^2 + |\nabla F|^2 + m^2 F^2 \right) \Big|_{x^0=0}. \quad (\text{C.10})$$

C.2. Explicit computations of section 5.5.2.3

We first define $g_R^s := E_\varphi(g_{\varphi,R}^s) + E_\pi(g_{\pi,R}^s) \in \mathfrak{H}$. Then, we compute

$$\begin{aligned} & \langle 0 | \Delta_{\Phi,0}^{it_1} \Delta_{\Phi,0}^{it_2} | 0 \rangle \\ &= \langle 0 | e^{i\alpha(s_1)} W_\varphi(g_{\varphi,R}^{s_1}) W_\pi(g_{\pi,R}^{s_1}) \Delta_0^{it_1} e^{i\alpha(s_2)} W_\varphi(g_{\varphi,R}^{s_2}) W_\pi(g_{\pi,R}^{s_2}) \Delta_0^{it_2} | 0 \rangle \\ &= e^{i\alpha(s_1)+i\alpha(s_2)} \langle 0 | W_\varphi(g_{\varphi,R}^{s_1}) W_\pi(g_{\pi,R}^{s_1}) e^{is_1 K_1} W_\varphi(g_{\varphi,R}^{s_2}) W_\pi(g_{\pi,R}^{s_2}) | 0 \rangle \\ &= e^{i\alpha(s_1)+i\alpha(s_2)-i\text{Im}\langle g_{\varphi,R}^{s_1} | g_{\pi,R}^{s_1} \rangle_{1\text{p}} - i\text{Im}\langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1\text{p}}} \langle 0 | W(g_R^{s_1}) e^{is_1 K_1} W(g_R^{s_2}) | 0 \rangle \\ &= e^{i\alpha(s_1)+i\alpha(s_2)-i\text{Im}\langle g_{\varphi,R}^{s_1} | g_{\pi,R}^{s_1} \rangle_{1\text{p}} - i\text{Im}\langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1\text{p}}} \langle 0 | W(g_R^{s_1}) e^{is_1 K_1} W(g_R^{s_2}) e^{-is_1 K_1} | 0 \rangle \\ &= e^{i\alpha(s_1)+i\alpha(s_2)-i\text{Im}\langle g_{\varphi,R}^{s_1} | g_{\pi,R}^{s_1} \rangle_{1\text{p}} - i\text{Im}\langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1\text{p}}} \langle 0 | W(g_R^{s_1}) W(u(\Lambda_1^{s_1}) g_R^{s_2}) | 0 \rangle \\ &= e^{i\alpha(s_1)+i\alpha(s_2)-i\text{Im}[\langle g_{\varphi,R}^{s_1} | g_{\pi,R}^{s_1} \rangle_{1\text{p}} + \langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1\text{p}} + \langle g_R^{s_1} | u(\Lambda_1^{s_1}) g_R^{s_2} \rangle_{1\text{p}}]} \langle 0 | W(g_R^{s_1} + u(\Lambda_1^{s_1}) g_R^{s_2}) | 0 \rangle \\ &= e^{i\alpha(s_1)+i\alpha(s_2)-i\text{Im}[\langle g_{\varphi,R}^{s_1} | g_{\pi,R}^{s_1} \rangle_{1\text{p}} + \langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1\text{p}} + \langle g_R^{s_1} | u(\Lambda_1^{s_1}) g_R^{s_2} \rangle_{1\text{p}}] - \frac{1}{2} \|g_R^{s_1} + u(\Lambda_1^{s_1}) g_R^{s_2}\|_{1\text{p}}^2}, \quad (\text{C.11}) \end{aligned}$$

and

$$\begin{aligned}
\langle 0 | \Delta_{\Phi,0}^{i(t_1+t_2)} | 0 \rangle &= \langle 0 | e^{i\alpha(s_1+s_2)} W_\varphi(g_{\varphi,R}^{s_1+s_2}) W_\pi(g_{\pi,R}^{s_1+s_2}) \Delta_0^{i(t_1+t_2)} | 0 \rangle \\
&= e^{i\alpha(s_1+s_2)} \langle 0 | W_\varphi(g_{\varphi,R}^{s_1+s_2}) W_\pi(g_{\pi,R}^{s_1+s_2}) | 0 \rangle \\
&= e^{i\alpha(s_1+s_2) - i\text{Im}\langle g_{\varphi,R}^{s_1+s_2} | g_{\pi,R}^{s_1+s_2} \rangle_{1\text{p}}} \langle 0 | W(g_R^{s_1+s_2}) | 0 \rangle \\
&= e^{i\alpha(s_1+s_2) - i\text{Im}\langle g_{\varphi,R}^{s_1+s_2} | g_{\pi,R}^{s_1+s_2} \rangle_{1\text{p}} - \frac{1}{2} \|g_R^{s_1+s_2}\|_{1\text{p}}^2}. \tag{C.12}
\end{aligned}$$

Taking $\frac{d}{ds_1} \Big|_{s_1=0}$ on both expressions above, we obtain

$$\begin{aligned}
\frac{d}{ds_1} \Big|_{s_1=0} \langle 0 | \Delta_{\Phi,0}^{it_1} \Delta_{\Phi,0}^{it_2} | 0 \rangle &= i\alpha'(0) - i \underbrace{\frac{d}{ds_1} \Big|_{s_1=0} \text{Im}\langle g_{\varphi,R}^{s_1} | g_{\pi,R}^{s_1} \rangle_{1\text{p}}}_{=0} \\
&\quad - i \frac{d}{ds_1} \Big|_{s_1=0} \text{Im}\langle g_R^{s_1} | u(\Lambda_1^{s_1}) g_R^{s_2} \rangle_{1\text{p}} - \frac{1}{2} \frac{d}{ds_1} \Big|_{s_1=0} \|g_R^{s_1} + u(\Lambda_1^{s_1}) g_R^{s_2}\|_{1\text{p}}^2 \\
&= i\alpha'(0) - \frac{d}{ds_1} \Big|_{s_1=0} \left(i\text{Im}\langle g_R^{s_1} | g_R^{s_2} \rangle_{1\text{p}} + \frac{1}{2} \|g_R^{s_1} + u(\Lambda_1^{s_1}) g_R^{s_2}\|_{1\text{p}}^2 \right), \tag{C.13}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{ds_1} \Big|_{s_1=0} \langle 0 | \Delta_{\Phi,0}^{i(t_1+t_2)} | 0 \rangle &= i\alpha'(s_2) - i \frac{d}{ds_1} \Big|_{s_1=0} \text{Im}\langle g_{\varphi,R}^{s_1+s_2} | g_{\pi,R}^{s_1+s_2} \rangle_{1\text{p}} - \frac{1}{2} \frac{d}{ds_1} \Big|_{s_1=0} \|g_R^{s_1+s_2}\|_{1\text{p}}^2 \\
&= i\alpha'(s_2) - i \frac{d}{ds_2} \text{Im}\langle g_{\varphi,R}^{s_2} | g_{\pi,R}^{s_2} \rangle_{1\text{p}} - \frac{1}{2} \frac{d}{ds_1} \Big|_{s_1=0} \|g_R^{s_1+s_2}\|_{1\text{p}}^2. \tag{C.14}
\end{aligned}$$

Matching separately the real and imaginary parts of these last two expressions, we arrive to formulas (5.139-5.140).

Expressions (5.141-5.144) follow from

$$\begin{aligned}
2 \text{Im}\langle g_R^{s_1} | g_R^{s_2} \rangle_{1\text{p}} &= 2 \text{Im}\langle g_{\varphi,R}^{s_1} + g_{\pi,R}^{s_1} | g_{\varphi,R}^{s_2} + g_{\pi,R}^{s_2} \rangle_{1\text{p}} \\
&= \int_{\Sigma} d^{d-1}x g_{\varphi}^{s_1}(\bar{x}) g_{\pi}^{s_2}(\bar{x}) - \int_{\Sigma} d^{d-1}x g_{\varphi}^{s_2}(\bar{x}) g_{\pi}^{s_1}(\bar{x}) \\
&= \int_{\Sigma} d^{d-1}x (f_{\varphi}^{s_1}(\bar{x}) - f_{\varphi}(\bar{x})) (f_{\pi}^{s_2}(\bar{x}) - f_{\pi}(\bar{x})) \\
&\quad - \int_{\Sigma} d^{d-1}x (f_{\varphi}^{s_2}(\bar{x}) - f_{\varphi}(\bar{x})) (f_{\pi}^{s_1}(\bar{x}) - f_{\pi}(\bar{x})) \\
&= \underbrace{\int_{\Sigma_R} d^{d-1}x f_{\varphi}(\bar{x}) f_{\pi}^{s_1}(\bar{x}) - \int_{\Sigma} d^{d-1}x f_{\varphi}^{s_1}(\bar{x}) f_{\pi}(\bar{x})}_{:=P(s_1)} \\
&\quad + \underbrace{\int_{\Sigma_R} d^{d-1}x f_{\varphi}^{s_1}(\bar{x}) f_{\pi}^{s_2}(\bar{x})}_{:=Q(s_1,s_2)} - \underbrace{\int_{\Sigma_R} d^{d-1}x f_{\varphi}^{s_2}(\bar{x}) f_{\pi}^{s_1}(\bar{x})}_{:=R(s_1,s_2)} + \gamma(s_2), \tag{C.15}
\end{aligned}$$

where the function γ includes all the s_1 -independent terms.

The function $P(s_1)$ is essentially the same as (C.2) in appendix C.1, with the difference that now the integration is over the region $\Sigma_R = \{\bar{x} \in \mathbb{R}^{d-1} : x^1 > 0\}$ instead of the whole space \mathbb{R}^{d-1} . Despite this, the final result is the same, and hence we get¹

$$\left. \frac{dP}{ds_1} \right|_{s_1=0} = \int_{\Sigma_R} d^{d-1}x x^1 \left(\left(\frac{\partial F}{\partial x^0} \right)^2 + (\nabla F)^2 + m^2 F^2 \right) \Big|_{x^0=0} =: \mathbf{S}. \quad (\text{C.16})$$

Now, we explicitly obtain the relations (5.146). Then, we compute

$$\begin{aligned} \left. \frac{dR}{ds_1} \right|_{s_1=0} &= \left. \frac{d}{ds_1} \right|_{s_1=0} \int_{\Sigma_R} d^{d-1}x \left(-\cosh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \sinh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right) F(\bar{x}^{s_1}) \\ &= \int_{\Sigma_R} d^{d-1}x \left(-\cosh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \sinh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right) \left(-x^1 \frac{\partial F}{\partial x^0}(\bar{x}) \right) \\ &= \int_{\Sigma_R} d^{d-1}x \left(-\frac{\partial F}{\partial x^0}(\bar{x}) \right) \left(-x^1 \cosh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + x^1 \sinh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right) \\ &= \left. \frac{d}{ds_2} \right|_{s_2=0} \int_{\Sigma_R} d^{d-1}x \left(-\frac{\partial F}{\partial x^0}(\bar{x}) \right) F(\bar{x}^{s_2}) \\ &= \left. \frac{d}{ds_2} \right|_{s_1=0} \int_{\Sigma_R} d^{d-1}x \left(-\cosh(s_1) \frac{\partial F}{\partial x^0}(\bar{x}^{s_1}) + \sinh(s_1) \frac{\partial F}{\partial x^1}(\bar{x}^{s_1}) \right) F(\bar{x}^{s_2}) \\ &= \left. \frac{d}{ds_2} \right|_{s_1=0} \int_{\Sigma_R} d^{d-1}x f_\varphi^{s_1}(\bar{x}) f_\pi^{s_2}(\bar{x}) = \left. \frac{dQ}{ds_2} \right|_{s_1=0}. \end{aligned} \quad (\text{C.17})$$

Similarly, we start with

$$\begin{aligned} \left. \frac{dQ}{ds_1} \right|_{s_1=0} &= \left. \frac{d}{ds_1} \right|_{s_1=0} \int_{\Sigma_R} d^{d-1}x f_\varphi^{s_1}(\bar{x}) f_\pi^{s_2}(\bar{x}) \\ &= \left. \frac{d}{ds_1} \right|_{s_1=0} \int_{\Sigma_R} d^{d-1}x \left(-\cosh(s_1) \frac{\partial F}{\partial x^0}(\bar{x}^{s_1}) + \sinh(s_1) \frac{\partial F}{\partial x^1}(\bar{x}^{s_1}) \right) F(\bar{x}^{s_2}) \\ &= \int_{\Sigma_R} d^{d-1}x \left(x^1 \frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}) + \frac{\partial F}{\partial x^1}(\bar{x}) \right) F(\bar{x}^{s_2}) \\ &= \int_{\Sigma_R} d^{d-1}x \left(x^1 (\nabla^2 - m^2) F(\bar{x}) + \frac{\partial F}{\partial x^1}(\bar{x}) \right) F(\bar{x}^{s_2}). \end{aligned} \quad (\text{C.18})$$

Then, we integrate the Laplacian term by parts

$$\begin{aligned} \left. \frac{dQ}{ds_1} \right|_{s_1=0} &= - \int_{\Sigma_R} d^{d-1}x x^1 m^2 F(\bar{x}) F(\bar{x}^{s_2}) - \int_{\Sigma} d^{d-1}x x^1 \nabla_\perp F(\bar{x}) \cdot \nabla_\perp F(\bar{x}^{s_2}) \\ &\quad - \int_{\Sigma_R} d^{d-1}x x^1 \frac{\partial F}{\partial x^1}(\bar{x}) \left(-\sinh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right). \end{aligned} \quad (\text{C.19})$$

¹In contrast with the computation of (C.2) in section C.1, now it appears a boundary term after the integration by parts. Fortunately, this term vanishes since the integrand is zero at the boundary of Σ_R .

After a second integration by parts, we get

$$\begin{aligned} \left. \frac{dQ}{ds_1} \right|_{s_1=0} &= \int_{\Sigma_R} d^{d-1}x x^1 F(\bar{x}) (\nabla_{\perp}^2 - m^2) F(\bar{x}^{s_2}) \\ &+ \int_{\Sigma_R} d^{d-1}x F(\bar{x}) \left(-\sinh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right) \\ &+ \int_{\Sigma_R} d^{d-1}x x^1 F(\bar{x}) \left(\sinh^2(s_2) \frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) \right. \\ &\left. - 2 \sinh(s_2) \cosh(s_2) \frac{\partial^2 F}{\partial x^0 \partial x^1}(\bar{x}^{s_2}) + \cosh^2(s_2) \frac{\partial^2 F}{(\partial x^1)^2}(\bar{x}^{s_2}) \right). \quad (\text{C.20}) \end{aligned}$$

Now, we form a Laplacian term in the first line and we use the equation of motion for F ,

$$\begin{aligned} \left. \frac{dQ}{ds_1} \right|_{s_1=0} &= \int_{\Sigma_R} d^{d-1}x x^1 F(\bar{x}) \frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) \\ &+ \int_{\Sigma_R} d^{d-1}x F(\bar{x}) \left(-\sinh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \cosh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right) \\ &+ \int_{\Sigma_R} d^{d-1}x x^1 F(\bar{x}) \left(\sinh^2(s_2) \frac{\partial^2 F}{(\partial x^0)^2}(\bar{x}^{s_2}) \right. \\ &\left. - 2 \sinh(s_2) \cosh(s_2) \frac{\partial^2 F}{\partial x^0 \partial x^1}(\bar{x}^{s_2}) + \sinh^2(s_2) \frac{\partial^2 F}{(\partial x^1)^2}(\bar{x}^{s_2}) \right). \quad (\text{C.21}) \end{aligned}$$

Finally, a straightforward computation shows that

$$\begin{aligned} \left. \frac{dQ}{ds_1} \right|_{s_1=0} &= \int_{\Sigma_R} d^{d-1}x \frac{d}{ds_2} \left(-\cosh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \sinh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right) F(\bar{x}) \\ &= \frac{d}{ds_2} \Big|_{s_1=0} \int_{\Sigma_R} d^{d-1}x \left(-\cosh(s_2) \frac{\partial F}{\partial x^0}(\bar{x}^{s_2}) + \sinh(s_2) \frac{\partial F}{\partial x^1}(\bar{x}^{s_2}) \right) F(\bar{x}^{s_1}) \\ &= \frac{d}{ds_2} \Big|_{s_1=0} \int_{\Sigma_R} d^{d-1}x f_{\varphi}^{s_2}(\bar{x}) f_{\pi}^{s_1}(\bar{x}) = \left. \frac{dR}{ds_2} \right|_{s_1=0}. \quad (\text{C.22}) \end{aligned}$$

Using (C.17) and (C.22), we arrive to (5.146).

C.3. Analytic continuation for $N(s)$

In order to show that formulas (5.153-5.154) hold, we need to explicitly obtain the analytic continuation of the function

$$N(s) = \frac{i}{2} (Q(0, s) - R(0, s)) - \frac{1}{2} \|g_R^s\|_{1p}^2, \quad (\text{C.23})$$

or more specifically, we need to show that there exists a continuous function $\tilde{N} : \mathbb{R} + i[0, 2\pi] \rightarrow \mathbb{C}$, analytic on $\mathbb{R} + i(0, 2\pi)$, such that

$$\tilde{N}(s + i0) = N(s) . \quad (\text{C.24})$$

To begin with, we notice that

$$\frac{i}{2}Q(0, s) = \frac{i}{2} \int_{x^1 > 0} d^{d-1}x f_\varphi(\bar{x}) f_\pi^s(\bar{x}) = i \operatorname{Im} \langle f_{\varphi,R} | f_{\pi,R}^s \rangle_{1\text{p}} , \quad (\text{C.25})$$

$$\frac{i}{2}R(0, s) = \frac{1}{2} \int_{x^1 > 0} d^{d-1}x f_\varphi^s(\bar{x}) f_\pi(\bar{x}) = i \operatorname{Im} \langle f_{\varphi,R}^s | f_{\pi,R} \rangle_{1\text{p}} , \quad (\text{C.26})$$

where the above expressions make sense regardless $f_{\pi,R}^s \notin \mathfrak{H}$. This is because

$$\langle f_{\varphi,R} | f_{\pi,R}^s \rangle_{1\text{p}} = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_{\bar{p}}} \hat{f}_{\varphi,R}(\bar{p})^* i\omega_{\bar{p}} \hat{f}_{\pi,R}^s(\bar{p}) = \frac{i}{2} \langle \hat{f}_{\varphi,R} | \hat{f}_{\pi,R}^s \rangle_{L^2} , \quad (\text{C.27})$$

which is convergent. The problem involving scalar products of split functions $f_{\varphi,R}^s$ and $f_{\pi,R}^s$ happens only when we try to compute the scalar product of two sharply cut test functions of the momentum operator, e.g.

$$\langle f_{\pi,R} | f_{\pi,R}^s \rangle_{1\text{p}} = \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_{\bar{p}}} \left(i\omega_{\bar{p}} \hat{f}_{\pi,R}(\bar{p}) \right)^* i\omega_{\bar{p}} \hat{f}_{\pi,R}^s(\bar{p}) = \frac{1}{2} \langle \hat{f}_{\pi,R} | \hat{f}_{\pi,R}^s \rangle_{H^{\frac{1}{2}}} , \quad (\text{C.28})$$

which is in general divergent. Such divergence comes from the non-continuity of the function $f_{\pi,R}(\bar{x}) = f_\pi(\bar{x}) \Theta(x^1)$ at $x^1 = 0$. To overcome this difficulty, we introduce a family of smooth functions (for $\epsilon > 0$)

$$f_{\varphi,R}^\epsilon(\bar{x}) := f_\varphi(\bar{x}) \Theta_\epsilon(x^1) \quad \text{and} \quad f_{\pi,R}^\epsilon(\bar{x}) := f_\pi(\bar{x}) \Theta_\epsilon(x^1) , \quad (\text{C.29})$$

where $\Theta_\epsilon \in C^\infty(\mathbb{R})$ is a regularized Heaviside function such that

$$\Theta_\epsilon(t) = \begin{cases} 0 & \text{if } t \leq \frac{\epsilon}{2} \\ 1 & \text{if } t \geq \epsilon \end{cases} . \quad (\text{C.30})$$

Then, we have that

$$f_{\varphi,R}^\epsilon(\bar{x}) \xrightarrow{\epsilon \rightarrow 0^+} f_{\varphi,R}(\bar{x}) \quad \text{and} \quad f_{\pi,R}^\epsilon(\bar{x}) \xrightarrow{\epsilon \rightarrow 0^+} f_{\pi,R}(\bar{x}) , \quad (\text{C.31})$$

where the above convergence must be in a sense that we will specify opportunely below. Before we get into such convergence issues, we notice that $f_{\varphi,R}^\epsilon, f_{\pi,R}^\epsilon \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ and

hence, the scalar product (C.28) is now well-defined. Then, we define the function

$$N^\epsilon(s) := i \operatorname{Im} \langle f_{\varphi,R}^\epsilon | f_{\pi,R}^{s,\epsilon} \rangle_{1\text{p}} - i \operatorname{Im} \langle f_{\varphi,R}^{s,\epsilon} | f_{\pi,R}^\epsilon \rangle_{1\text{p}} - \frac{1}{2} \|g_R^{s,\epsilon}\|_{1\text{p}}^2, \quad (\text{C.32})$$

which is just the regularized version of (C.23). In the next subsection, we will show that $N^\epsilon(s) \rightarrow N(s)$ when $\epsilon \rightarrow 0^+$. Expression (C.32) can be rewritten as

$$\begin{aligned} N^\epsilon(s) &= i \operatorname{Im} \langle f_{\varphi,R}^\epsilon | f_{\pi,R}^{s,\epsilon} \rangle_{1\text{p}} + i \operatorname{Im} \langle f_{\pi,R}^\epsilon | f_{\varphi,R}^{s,\epsilon} \rangle_{1\text{p}} - \frac{1}{2} \langle f_R^\epsilon - f_R^{s,\epsilon} | f_R^\epsilon - f_R^{s,\epsilon} \rangle_{1\text{p}} \\ &= \langle f_R^\epsilon | f_R^{s,\epsilon} \rangle_{1\text{p}} - \frac{1}{2} \langle f_R^\epsilon | f_R^\epsilon \rangle_{1\text{p}} - \frac{1}{2} \langle f_R^{s,\epsilon} | f_R^{s,\epsilon} \rangle_{1\text{p}} = \langle f_R^{-\frac{s}{2},\epsilon} | f_R^{\frac{s}{2},\epsilon} \rangle_{1\text{p}} - \langle f_R^\epsilon | f_R^\epsilon \rangle_{1\text{p}} \\ &= \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_{\bar{p}}} \left[\left(\hat{f}_{\varphi,R}^{-\frac{s}{2},\epsilon} + i\omega_{\bar{p}} \hat{f}_{\pi,R}^{-\frac{s}{2},\epsilon} \right)^* \left(\hat{f}_{\varphi,R}^{\frac{s}{2},\epsilon} + i\omega_{\bar{p}} \hat{f}_{\pi,R}^{\frac{s}{2},\epsilon} \right) \right. \\ &\quad \left. - \left(\hat{f}_{\varphi,R}^\epsilon + i\omega_{\bar{p}} \hat{f}_{\pi,R}^\epsilon \right)^* \left(\hat{f}_{\varphi,R}^\epsilon + i\omega_{\bar{p}} \hat{f}_{\pi,R}^\epsilon \right) \right], \end{aligned} \quad (\text{C.33})$$

where in the penultimate line we have used that $f_R^{s_1+s_2,\epsilon} = u(\Lambda_1^{s_2}) f_R^{s_1,\epsilon}$ for all $s_1, s_2 \in \mathbb{R}$. For a moment, let assume that this last expression converges to

$$\begin{aligned} N(s) &= \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_{\bar{p}}} \left[\left(\hat{f}_{\varphi,R}^{-\frac{s}{2}} + i\omega_{\bar{p}} \hat{f}_{\pi,R}^{-\frac{s}{2}} \right)^* \left(\hat{f}_{\varphi,R}^{\frac{s}{2}} + i\omega_{\bar{p}} \hat{f}_{\pi,R}^{\frac{s}{2}} \right) \right. \\ &\quad \left. - \left(\hat{f}_{\varphi,R} + i\omega_{\bar{p}} \hat{f}_{\pi,R} \right)^* \left(\hat{f}_{\varphi,R} + i\omega_{\bar{p}} \hat{f}_{\pi,R} \right) \right], \end{aligned} \quad (\text{C.34})$$

when $\epsilon \rightarrow 0^+$. We will prove this in the next subsection. The second term of the above integrand is independent on s and hence its analytic continuation is trivial. Let us then focus in the first term. Using the Poincaré covariance and causality of the Klein-Gordon equation, it is not difficult to show that

$$\hat{f}_{\varphi,R}^s(\bar{p}) + i\omega_{\bar{p}} \hat{f}_{\pi,R}^s(\bar{p}) = \hat{f}_{\varphi,R}(\Lambda_1^s \bar{p}) + i\Lambda_1^s \omega_{\bar{p}} \hat{f}_{\pi,R}(\Lambda_1^s \bar{p}), \quad (\text{C.35})$$

where $\Lambda_1^s \bar{p} = (p^1 \cosh(s) - \omega_{\bar{p}} \sinh(s), \bar{p}_\perp)$ and $\Lambda_1^s \omega_{\bar{p}} = \omega_{\bar{p}} \cosh(s) - p^1 \sinh(s)$. Then, the first integrand term of (C.34) becomes

$$\begin{aligned} &\left(\hat{f}_{\varphi,R}^{-\frac{s}{2},\epsilon}(\bar{p}) + i\omega_{\bar{p}} \hat{f}_{\pi,R}^{-\frac{s}{2},\epsilon}(\bar{p}) \right)^* \left(\hat{f}_{\varphi,R}^{\frac{s}{2},\epsilon}(\bar{p}) + i\omega_{\bar{p}} \hat{f}_{\pi,R}^{\frac{s}{2},\epsilon}(\bar{p}) \right) \\ &= \int_{\mathbb{R}^{2(d-1)}} d^{d-1}x d^{d-1}y (f_{\varphi,R}(\bar{x}) - i\omega_{\bar{p}} f_{\pi,R}(\bar{x})) (f_{\varphi,R}(\bar{y}) + i\omega_{\bar{p}} f_{\pi,R}(\bar{y})) e^{i\Lambda^{-\frac{s}{2}}(\bar{p}) \cdot \bar{x} - i\Lambda^{\frac{s}{2}}(\bar{p}) \cdot \bar{y}}, \end{aligned} \quad (\text{C.36})$$

where $-i\Lambda^{\frac{s}{2}}(\bar{p}) \cdot \bar{y} = -i\left(-\sinh\left(\frac{s}{2}\right)\omega_{\bar{p}} + \cosh\left(\frac{s}{2}\right)p^1\right)y^1 - i\bar{p}_\perp \cdot \bar{y}_\perp$, and equivalently for $i\Lambda^{-\frac{s}{2}}(\bar{p}) \cdot \bar{x}$. Then, we have that

$$\begin{aligned} -i\left(-\sinh\left(\frac{s}{2}\right)\omega_{\bar{p}} + \cosh\left(\frac{s}{2}\right)p^1\right)y^1 &\xrightarrow{s \rightarrow s+i\sigma} -i\left(-\sinh\left(\frac{s+i\sigma}{2}\right)\omega_{\bar{p}} + \cosh\left(\frac{s+i\sigma}{2}\right)p^1\right)y^1 \\ &= -i\left(-\sinh\left(\frac{s}{2}\right)\omega_{\bar{p}} + \cosh\left(\frac{s}{2}\right)p^1\right)y^1 \cos\left(\frac{\sigma}{2}\right) \\ &\quad - \underbrace{\left(\cosh\left(\frac{s}{2}\right)\omega_{\bar{p}} - \sinh\left(\frac{s}{2}\right)p^1\right)y^1}_{\geq m} \sin\left(\frac{\sigma}{2}\right), \end{aligned} \quad (\text{C.37})$$

where the second term provides an exponential dumping in equation (C.36) when $\sigma \in (0, 2\pi)$ because $\text{supp}(f_{\varphi,R}), \text{supp}(f_{\pi,R}) \subset \Sigma_R$. Equivalently, it can be shown that $i\Lambda^{-\frac{s}{2}}(\bar{p}) \cdot \bar{x}$ also provides an exponential dumping for $\sigma \in (0, 2\pi)$. Hence, we have that

$$\tilde{N}(s+i\sigma) \text{ is an analytic function for } s+i\sigma \in \mathbb{R} + i(0, 2\pi). \quad (\text{C.38})$$

Looking at expressions (C.34) and (C.36), it is easy to determine that

$$\lim_{\sigma \rightarrow 2\pi^-, s=0} \tilde{N}(s+i\sigma) = 0. \quad (\text{C.39})$$

C.3.1. Convergence of $N^\epsilon(s)$

In order to show that expression (C.34) holds, we need to prove the following two limits

$$N^\epsilon(s) \xrightarrow{\epsilon \rightarrow 0^+} N(s) = \frac{i}{2} (Q(0, s) - R(0, s)) - \frac{1}{2} \|g_R^s\|_{\mathfrak{H}}^2, \quad (\text{C.40})$$

$$\begin{aligned} N^\epsilon(s) &\xrightarrow{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{2\omega_{\bar{p}}} \left[\left(\hat{f}_{\varphi,R}^{-\frac{s}{2}} + i\omega_{\bar{p}} \hat{f}_{\pi,R}^{-\frac{s}{2}} \right)^* \left(\hat{f}_{\varphi,R}^{\frac{s}{2}} + i\omega_{\bar{p}} \hat{f}_{\pi,R}^{\frac{s}{2}} \right) \right. \\ &\quad \left. - \left(\hat{f}_{\varphi,R} + i\omega_{\bar{p}} \hat{f}_{\pi,R} \right)^* \left(\hat{f}_{\varphi,R} + i\omega_{\bar{p}} \hat{f}_{\pi,R} \right) \right]. \end{aligned} \quad (\text{C.41})$$

To do this, we must precise in which sense the functions $f_{\varphi,R}^{s,\epsilon}, f_{\pi,R}^{s,\epsilon}$ converge in (C.31). To start, we choose the following smooth step function (C.30)

$$\Theta_\epsilon(t) = \begin{cases} 0 & \text{if } t \leq \frac{\epsilon}{2}, \\ \left[\exp\left(\frac{\epsilon(t-\frac{3\epsilon}{4})}{(t-\frac{3\epsilon}{4})^2 - (\frac{\epsilon}{4})^2}\right) + 1 \right]^{-1} & \text{if } \frac{\epsilon}{2} < t < \epsilon, \\ 1 & \text{if } t \geq \epsilon. \end{cases} \quad (\text{C.42})$$

Let us first focus in the limit (C.41). Looking back to (C.33), we can rewrite the r.h.s. of that expression as

$$\begin{aligned}
N^\epsilon(s) &= \langle f_R^{-\frac{s}{2}, \epsilon} | f_R^{\frac{s}{2}, \epsilon} \rangle_{1\mathbb{P}} - \langle f_R^\epsilon | f_R^\epsilon \rangle_{1\mathbb{P}} = \langle f_R^{-\frac{s}{2}, \epsilon} | f_R^{\frac{s}{2}, \epsilon} - f_R^\epsilon + f_R^\epsilon \rangle_{1\mathbb{P}} - \langle f_R^\epsilon | f_R^\epsilon \rangle_{1\mathbb{P}} \\
&= \langle f_{\varphi, R}^{-\frac{s}{2}, \epsilon} | f_{\varphi, R}^{\frac{s}{2}, \epsilon} - f_{\varphi, R}^\epsilon \rangle_{1\mathbb{P}} + \langle f_{\varphi, R}^{-\frac{s}{2}, \epsilon} | f_{\pi, R}^{\frac{s}{2}, \epsilon} - f_{\pi, R}^\epsilon \rangle_{1\mathbb{P}} + \langle f_{\varphi, R}^{-\frac{s}{2}, \epsilon} - f_{\varphi, R}^\epsilon | f_{\varphi, R}^\epsilon \rangle_{1\mathbb{P}} \\
&\quad + \langle f_{\pi, R}^{-\frac{s}{2}, \epsilon} - f_{\pi, R}^\epsilon | f_{\varphi, R}^\epsilon \rangle_{1\mathbb{P}} + \langle f_{\pi, R}^{-\frac{s}{2}, \epsilon} | f_{\varphi, R}^{\frac{s}{2}, \epsilon} - f_{\varphi, R}^\epsilon \rangle_{1\mathbb{P}} + \langle f_{\varphi, R}^{-\frac{s}{2}, \epsilon} - f_{\varphi, R}^\epsilon | f_{\pi, R}^\epsilon \rangle_{1\mathbb{P}} \\
&\quad + \underbrace{\langle f_{\pi, R}^{-\frac{s}{2}, \epsilon} | f_{\pi, R}^{\frac{s}{2}, \epsilon} - f_{\pi, R}^\epsilon \rangle_{1\mathbb{P}}}_{\otimes} + \underbrace{\langle f_{\pi, R}^{-\frac{s}{2}, \epsilon} - f_{\pi, R}^\epsilon | f_{\pi, R}^\epsilon \rangle_{1\mathbb{P}}}_{\otimes}. \tag{C.43}
\end{aligned}$$

It is not difficult to see that

$$f_{\varphi, R}^{s, \epsilon} \xrightarrow{\epsilon \rightarrow 0^+} f_{\varphi, R}^s \quad \text{and} \quad f_{\pi, R}^{s, \epsilon} \xrightarrow{\epsilon \rightarrow 0^+} f_{\pi, R}^s, \quad \text{in } L^2(\mathbb{R}^{d-1}), \tag{C.44}$$

which implies that all terms in (C.43) are convergent, except, perhaps, those pointed by \otimes . Now, we concentrate in those remaining terms, for example

$$\langle f_{\pi, R}^{-\frac{s}{2}, \epsilon} | f_{\pi, R}^{\frac{s}{2}, \epsilon} - f_{\pi, R}^\epsilon \rangle_{1\mathbb{P}} = \frac{1}{2} \int_{\mathbb{R}^{d-1}} d^{d-1}p \hat{f}_{\pi, R}^{-\frac{s}{2}, \epsilon}(\bar{p}) \left(\hat{f}_{\pi, R}^{-\frac{s}{2}, \epsilon}(\bar{p}) - \hat{f}_{\pi, R}^\epsilon(\bar{p}) \right) \omega_{\bar{p}}. \tag{C.45}$$

The convergence of (C.45) is guaranteed by the fact that

$$f_{\pi, R}^\epsilon - f_{\pi, R}^{-\frac{s}{2}, \epsilon} \xrightarrow{\epsilon \rightarrow 0^+} f_{\pi, R} - f_{\pi, R}^{-\frac{s}{2}} \quad \text{in } H^1(\mathbb{R}^{d-1}), \tag{C.46}$$

$$\begin{aligned} &\Downarrow \\ \left(\hat{f}_{\pi, R}^\epsilon - \hat{f}_{\pi, R}^{-\frac{s}{2}, \epsilon} \right) \omega_{\bar{p}} &\xrightarrow{\epsilon \rightarrow 0^+} \left(\hat{f}_{\pi, R} - \hat{f}_{\pi, R}^{-\frac{s}{2}} \right) \omega_{\bar{p}} \quad \text{in } L^2(\mathbb{R}^{d-1}). \end{aligned} \tag{C.47}$$

In order to prove (C.46), we remember that $f_{\pi, R}(\bar{x}) - f_{\pi, R}^s(\bar{x}) = g_\pi^s(\bar{x}) \Theta(x^1)$, with $g_\pi^s \in \mathcal{S}(\mathbb{R}^{d-1}, \mathbb{R})$ and $g_\pi^s|_{x^1=0} = 0$. Then, the following lemma ensures (C.46).

Lemma C.1. Let $g \in \mathcal{S}(\mathbb{R}^n)$ with $g|_{x^1=0} = 0$, $g_R(\bar{x}) = g(\bar{x}) \Theta(x^1)$ and $g_R^\epsilon(\bar{x}) = g(\bar{x}) \Theta_\epsilon(x^1)$ with Θ_ϵ as (C.42). Then, $g_R \in H^1(\mathbb{R}^n)$ and $g_R^\epsilon \xrightarrow{\epsilon \rightarrow 0^+} g_R$ in $H^1(\mathbb{R}^n)$.

Proof. The fact that $g_R \in H^1(\mathbb{R}^n)$ is guaranteed by lemma 5.11. Here, we prove the convergence for $n = 1$. The generalization to $n > 1$ is straightforward. Since g_R and g_R^ϵ satisfy the hypothesis of the lemma B.2, their weak derivatives coincide with their pointwise

derivatives, and hence

$$\begin{aligned}
\|g_R^\epsilon - g_R\|_{H^1}^2 &= \int_{-\infty}^{+\infty} dx |g(x) \Theta_\epsilon(x) - g(x) \Theta(x)|^2 + \int_{-\infty}^{+\infty} dx |\partial_x [g(x) \Theta_\epsilon(x) - g(x) \Theta(x)]|^2 \\
&\leq \int_{-\infty}^{+\infty} dx |g(x)|^2 |\Theta_\epsilon(x) - \Theta(x)|^2 + \int_{-\infty}^{+\infty} dx |g'(x)|^2 |\Theta_\epsilon(x) - \Theta(x)|^2 \\
&\quad + \int_{-\infty}^{+\infty} dx |g(x)|^2 |\Theta'_\epsilon(x)|^2 + 2 \int_{-\infty}^{+\infty} dx |g(x)| |g'(x)| |\Theta_\epsilon(x) - \Theta(x)| |\Theta'_\epsilon(x)| \\
&\leq \int_{\frac{\epsilon}{2}}^\epsilon dx (|g(x)|^2 + |g'(x)|^2) + \int_{\frac{\epsilon}{2}}^\epsilon dx |g(x)|^2 |\Theta'_\epsilon(x)|^2 \\
&\quad + 2 \int_{\frac{\epsilon}{2}}^\epsilon dx |g(x)| |g'(x)| |\Theta'_\epsilon(x)|. \tag{C.48}
\end{aligned}$$

We notice that since $g \in C^\infty(\mathbb{R})$ and $g(0) = 0$, by Taylor's theorem, we have that $g(x) = g'(0)x + r(x)x$, with $r(x) \xrightarrow{x \rightarrow 0} 0$ and $r \in C^\infty(\mathbb{R})$. We also have that $\max_{x \in \mathbb{R}} |\Theta'_\epsilon(x)| = \frac{4}{\epsilon}$, which follows from the definition of that function. Then, using the above properties and assuming $0 < \epsilon \leq 1$, we have

$$\begin{aligned}
\|g_R^\epsilon - g_R\|_{H^1}^2 &\leq \max_{x \in [0,1]} (|g(x)|^2 + |g'(x)|^2) \int_{\frac{\epsilon}{2}}^\epsilon dx + \max_{x \in [0,1]} |g'(0) + r(x)|^2 \frac{16}{\epsilon^2} \int_{\frac{\epsilon}{2}}^\epsilon dx x^2 \\
&\quad + \max_{x \in [0,1]} |g'(x)| \max_{x \in [0,1]} |g'(0) + r(x)| \frac{8}{\epsilon} \int_{\frac{\epsilon}{2}}^\epsilon dx x \\
&\leq \max_{x \in [0,1]} (|g(x)|^2 + |g'(x)|^2) \frac{\epsilon}{2} + \max_{x \in [0,1]} |g'(0) + r(x)|^2 \frac{14}{3} \epsilon \\
&\quad + \max_{x \in [0,1]} |g'(x)| \max_{x \in [0,1]} |g'(0) + r(x)| 3\epsilon \xrightarrow{\epsilon \rightarrow 0^+} 0. \tag{C.49}
\end{aligned}$$

□

Then, we have that all terms in (C.43) converge. By continuity of the inner product, the limit of (C.43) is just this same expression but evaluated at $\epsilon = 0$, which coincides with the l.h.s of (C.44).

We can use the same arguments to prove the limit (C.40). The first two terms of (C.32) are convergent due to (C.44), and the remaining term is also convergent due to (C.46) and (C.47). Then, by the continuity of the inner product, we have that

$$N^\epsilon(s) \xrightarrow{\epsilon \rightarrow 0^+} N(s) = \frac{i}{2} (Q(0, s) - R(0, s)) - \frac{1}{2} \|g_R^s\|_{1p}^2. \tag{C.50}$$

Finally, expression (C.34) holds.

Appendix D

Constructing the regular representation

In this appendix we show how to construct the endomorphism corresponding to the regular representation of the group. We follow the constructions developed in [72]. The regular representation is defined as the direct sum of all its irreducible representations, each one appearing a number of times equal to its dimension

$$\rho_{reg} = \bigoplus_{\sigma \in \hat{G}} d_{\sigma} \rho_{\sigma}. \quad (\text{D.1})$$

By constructing such a reducible representation, we mean to provide a set of charged intertwiners \mathcal{I}_{σ}^j ($j = 1, \dots, d_{\sigma}$) satisfying¹

$$\mathcal{I}_{\sigma}^j : \iota \rightarrow \rho_{reg}, \quad (\text{D.2})$$

$$\mathcal{I}_{\sigma}^{j\dagger} \mathcal{I}_{\nu}^k = \delta_{jk} \delta_{\sigma\nu}, \quad (\text{D.3})$$

$$\sum_{\sigma \in \hat{G}} \sum_{j=1}^{d_{\sigma}} \mathcal{I}_{\sigma}^j \mathcal{I}_{\sigma}^{j\dagger} = \mathbf{1}. \quad (\text{D.4})$$

The first relation says that all \mathcal{I}_{σ}^j intertwine the vacuum representation ι to the regular one ρ_{reg} . Equivalently, $\mathcal{I}_{\sigma}^j|0\rangle$ is a state that transforms under the regular representation of the group. The second and third relations ensure that the regular endomorphism can be explicitly written as

$$\rho_{reg}(A) := \sum_{\sigma \in \hat{G}} \sum_{j=1}^{d_{\sigma}} \mathcal{I}_{\sigma}^j A \mathcal{I}_{\sigma}^{j\dagger}, \quad A \in \mathcal{A}. \quad (\text{D.5})$$

¹An intertwiner from one representation ρ to another σ is an operator $T \in \mathcal{A}$ satisfying $T\rho = \sigma T$, and it is usually denoted by $T : \rho \rightarrow \sigma$.

More importantly for us is that such relations allow to construct the finite dimensional algebra spanned by the elements

$$A = \sum_{\sigma, \nu \in \hat{G}} \sum_{j, k=1}^{d_\sigma} a_{\sigma j, \nu k} \mathcal{I}_\sigma^j \mathcal{I}_\nu^{k\dagger}, \quad a_{\sigma j, \nu k} \in \mathbb{C}. \quad (\text{D.6})$$

This algebra is used in chapter 7 to find lower bounds for the entropic order parameter.

There are two possible avenues to construct such a space of intertwiners. The first is to use the method described in the text for constructing the irreducible sectors. This approach requires to have some operator O_{reg} that takes us from the vacuum to the given charged sector, in this case the, regular one. This approach is quite sensible and physical when such operators can be found easily. For example, in gauge theories, it is simple to consider Wilson lines in any given representation.

There is also a complementary approach, that only requires knowledge of the charged intertwiners associated with the irreducible sectors $\sigma \in \hat{G}$. These are the operators V_σ^j ($j = 1, \dots, d_\sigma$) that are described in chapter 7. They satisfy

$$V_\sigma^j : \iota \rightarrow \rho_s, \quad (\text{D.7})$$

$$V_\sigma^{j\dagger} V_\sigma^k = \delta_{jk}, \quad (\text{D.8})$$

$$\sum_{j=1}^{d_\sigma} V_\sigma^j V_\sigma^{j\dagger} = \mathbf{1}, \quad (\text{D.9})$$

$$\rho_\sigma(A) = \sum_{j=1}^{d_\sigma} V_\sigma^j A V_\sigma^{j\dagger}, \quad A \in \mathcal{A}. \quad (\text{D.10})$$

Now, the regular representation as defined above in (D.1), implies the existence of partial isometries ω_σ^j ($j = 1, \dots, d_\sigma$), satisfying the following properties

$$\omega_\sigma^j : \rho_\sigma \rightarrow \rho_{reg}, \quad (\text{D.11})$$

$$\omega_\sigma^{j\dagger} \omega_\nu^k = \delta_{jk} \delta_{\sigma\nu}, \quad (\text{D.12})$$

$$\sum_{j=1}^{d_\sigma} \omega_\sigma^j \omega_\sigma^{j\dagger} = \mathbf{1}, \quad (\text{D.13})$$

$$\sum_{j=1}^{d_\sigma} \omega_\sigma^j \rho_s(A) \omega_\sigma^{j\dagger} (\omega_s^i)^\dagger = \rho_{reg}(A), \quad A \in \mathcal{A}. \quad (\text{D.14})$$

Furthermore, these operators were explicitly constructed in [72], with the help of particular charged intertwiner to the regular endomorphism $v : \iota \rightarrow \rho_{reg}$. Its explicit construction might be cumbersome, but its existence is guaranteed for finite groups [72]. Assuming we

have such an operator, in [72] it was shown that there is an anti-isomorphism, such that

$$\omega_\sigma^j = |G| \varepsilon(v V_\sigma^{j\dagger}), \quad \text{for all } \sigma \in \hat{G} \text{ and } j = 1, \dots, d_\sigma, \quad (\text{D.15})$$

where ε is the conditional expectation and $|G|$ is the order of the group. Then, given V_σ^j and v , we can construct all the operators ω_σ^j . It is simple now to find the charged operators of the regular representation. Since $V_\sigma^j : \iota \rightarrow \rho_\sigma$ and $\omega_\sigma^j : \rho_\sigma \rightarrow \rho_{reg}$, it is clear that

$$\omega_\sigma^j V_\sigma^j : \iota \rightarrow \rho_{reg}. \quad (\text{D.16})$$

Moreover, given (D.10) and (D.14), it is simple to verify that relations (D.4) hold for $\mathcal{I}_\sigma^j := \omega_\sigma^j V_\sigma^j$.

Bibliography

- [1] Bogolyubov, N. N., Logunov, A. A., Oksak, A. I., Todorov, I. T. General principles of quantum field theory. Dordrecht, Netherlands: Kluwer (1990) 694 p. (Mathematical physics and applied mathematics, 10), 1990. [1](#), [17](#), [21](#), [22](#), [252](#)
- [2] Haag, R. Local Quantum physics: Fields, Particles, Algebras. Berlin, Germany: Springer-Verlag, 1992. [1](#), [4](#), [16](#), [18](#), [25](#), [28](#), [40](#), [51](#), [52](#), [56](#), [198](#), [209](#), [236](#), [281](#), [285](#)
- [3] Kitaev, A., Preskill, J. Topological entanglement entropy. *Phys. Rev. Lett.*, **96**, 110404, 2006. [3](#), [227](#)
- [4] Levin, M., Wen, X.-G. Detecting Topological Order in a Ground State Wave Function. *Phys. Rev. Lett.*, **96**, 110405, 2006. [3](#), [227](#)
- [5] Calabrese, P., Cardy, J. Entanglement entropy and conformal field theory. *J. Phys.*, **A42**, 504005, 2009. [3](#), [281](#)
- [6] Holzhey, C., Larsen, F., Wilczek, F. Geometric and renormalized entropy in conformal field theory. *Nucl. Phys.*, **B424**, 443–467, 1994.
- [7] Vidal, G., Latorre, J. I., Rico, E., Kitaev, A. Entanglement in quantum critical phenomena. *Phys. Rev. Lett.*, **90**, 227902, 2003.
- [8] Fradkin, E., Moore, J. E. Entanglement entropy of 2D conformal quantum critical points: hearing the shape of a quantum drum. *Phys. Rev. Lett.*, **97**, 050404, 2006. [3](#)
- [9] Ryu, S., Takayanagi, T. Holographic derivation of entanglement entropy from AdS/CFT. *Phys. Rev. Lett.*, **96**, 181602, 2006. [3](#)
- [10] Jafferis, D. L., Lewkowycz, A., Maldacena, J., Suh, S. J. Relative entropy equals bulk relative entropy. *JHEP*, **06**, 004, 2016. [3](#), [289](#)
- [11] Faulkner, T., Lewkowycz, A., Maldacena, J. Quantum corrections to holographic entanglement entropy. *JHEP*, **11**, 074, 2013.
- [12] Aharony, O., Gubser, S. S., Maldacena, J. M., Ooguri, H., Oz, Y. Large N field theories, string theory and gravity. *Phys. Rept.*, **323**, 183–386, 2000.

- [13] Maldacena, J. M. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, **38**, 1113–1133, 1999. [Adv. Theor. Math. Phys.2,231(1998)]. [3](#)
- [14] Van Raamsdonk, M. Building up spacetime with quantum entanglement II: It from BC-bit, 2018. [3](#)
- [15] Wolf, M. M., Verstraete, F., Hastings, M. B., Cirac, J. I. Area Laws in Quantum Systems: Mutual Information and Correlations. *Phys. Rev. Lett.*, **100** (7), 070502, 2008. [3](#), [51](#), [81](#)
- [16] Srednicki, M. Entropy and area. *Phys. Rev. Lett.*, **71**, 666–669, 1993.
- [17] Eisert, J., Cramer, M., Plenio, M. B. Area laws for the entanglement entropy - a review. *Rev. Mod. Phys.*, **82**, 277–306, 2010. [3](#), [81](#)
- [18] Bombelli, L., Koul, R. K., Lee, J., Sorkin, R. D. A Quantum Source of Entropy for Black Holes. *Phys. Rev.*, **D34**, 373–383, 1986. [3](#)
- [19] Susskind, L., Uglum, J. Black hole entropy in canonical quantum gravity and superstring theory. *Phys. Rev.*, **D50**, 2700–2711, 1994.
- [20] Solodukhin, S. N. Entanglement entropy of black holes. *Living Rev. Rel.*, **14**, 8, 2011. [3](#)
- [21] Bratteli, O., Robinson, D. W. Operator Algebras and Quantum Statistical Mechanics 1: C* and W* Algebras, Symmetry Groups, Decomposition of States. Springer-Verlag New York), 1979. [4](#), [11](#), [12](#), [15](#), [16](#), [20](#), [23](#), [24](#), [51](#), [52](#), [53](#)
- [22] Faulkner, T., Lewkowycz, A. Bulk locality from modular flow. *JHEP*, **07**, 151, 2017. [4](#)
- [23] Faulkner, T., Li, M., Wang, H. A modular toolkit for bulk reconstruction. *JHEP*, **04**, 119, 2019. [4](#)
- [24] Araki, H. Relative Entropy of States of von Neumann Algebras, 1974. [4](#), [57](#), [58](#)
- [25] Araki, H. Relative entropy of states of von Neumann algebras II. *Publ. RIMS Kyoto Univ.*, **13**, 173–192, 1977. [4](#), [58](#), [60](#)
- [26] Casini, H., Huerta, M. A Finite entanglement entropy and the c-theorem. *Phys. Lett.*, **B600**, 142–150, 2004. [4](#), [83](#), [181](#), [261](#)
- [27] Casini, H., Huerta, M. A c-theorem for the entanglement entropy. *J. Phys.*, **A40**, 7031–7036, 2007.

- [28] Myers, R. C., Sinha, A. Seeing a c-theorem with holography. *Phys. Rev.*, **D82**, 046006, 2010.
- [29] Casini, H., Huerta, M. On the RG running of the entanglement entropy of a circle. *Phys. Rev.*, **D85**, 125016, 2012.
- [30] Casini, H., Landea, I. S., Torroba, G. The g-theorem and quantum information theory. *JHEP*, **10**, 140, 2016.
- [31] Casini, H., Testé, E., Torroba, G. Markov Property of the Conformal Field Theory Vacuum and the a Theorem. *Phys. Rev. Lett.*, **118** (26), 261602, 2017.
- [32] Casini, H., Teste, E., Torroba, G. All the entropies on the light-cone, 2018. [4](#), [83](#), [88](#)
- [33] Bisognano, J. J., Wichmann, E. H. On the Duality Condition for a Hermitian Scalar Field. *J. Math. Phys.*, **16**, 985–1007, 1975. [4](#), [75](#)
- [34] Hislop, P. D., Longo, R. Modular Structure of the Local Algebras Associated With the Free Massless Scalar Field Theory. *Commun. Math. Phys.*, **84**, 71, 1982. [4](#), [76](#), [157](#)
- [35] Casini, H., Huerta, M., Myers, R. C. Towards a derivation of holographic entanglement entropy. *JHEP*, **05**, 036, 2011. [4](#), [76](#), [83](#), [157](#), [273](#)
- [36] Casini, H., Huerta, M. Reduced density matrix and internal dynamics for multicomponent regions. *Class. Quant. Grav.*, **26**, 185005, 2009. [4](#), [6](#), [133](#), [137](#), [138](#), [143](#), [146](#), [147](#), [170](#), [260](#), [287](#)
- [37] Arias, R., Blanco, D., Casini, H., Huerta, M. Local temperatures and local terms in modular Hamiltonians. *Phys. Rev.*, **D95** (6), 065005, 2017. [4](#), [76](#), [179](#)
- [38] Arias, R., Casini, H., Huerta, M., Pontello, D. Anisotropic Unruh temperatures. *Phys. Rev.*, **D96** (10), 105019, 2017. [4](#), [138](#), [179](#), [264](#), [267](#), [287](#), [288](#)
- [39] Arias, R. E., Casini, H., Huerta, M., Pontello, D. Entropy and modular Hamiltonian for a free chiral scalar in two intervals. *Phys. Rev.*, **D98** (12), 125008, 2018. [4](#), [95](#), [96](#), [265](#)
- [40] Blanco, D., Pérez-Nadal, G. Modular Hamiltonian of a chiral fermion on the torus. *Phys. Rev.*, **D100** (2), 025003, 2019. [4](#)
- [41] Streater, R. F., Wightman, A. S. PCT, spin and statistics, and all that. Redwood City, USA: Addison-Wesley (1989) 207 p. (Advanced book classics), 1989. [4](#), [29](#), [30](#), [31](#), [194](#)

- [42] Casini, H., Grillo, S., Pontello, D. Relative entropy for coherent states from Araki formula. *Phys. Rev.*, **D99** (12), 125020, 2019. [5](#), [102](#)
- [43] Longo, R. Index of subfactors and statistics of quantum fields. I. *Commun. Math. Phys.*, **126**, 217–247, 1989. [6](#), [211](#), [233](#), [261](#)
- [44] Longo, R. Index of subfactors and statistics of quantum fields. 2: Correspondences, braid group statistics and Jones polynomial. *Commun. Math. Phys.*, **130**, 285–309, 1990. [6](#), [233](#)
- [45] Doplicher, S., Haag, R., Roberts, J. E. Fields, observables and gauge transformations I. *Commun. Math. Phys.*, **13**, 1–23, 1969. [6](#), [190](#), [198](#), [208](#), [281](#)
- [46] Doplicher, S., Haag, R., Roberts, J. E. Fields, observables and gauge transformations II. *Commun. Math. Phys.*, **15**, 173–200, 1969. [34](#), [189](#), [202](#), [209](#), [281](#)
- [47] Doplicher, S., Haag, R., Roberts, J. E. Local observables and particle statistics I. *Commun. Math. Phys.*, **23**, 199–230, 1971. [199](#), [204](#)
- [48] Doplicher, S., Roberts, J. E. Fields, statistics and non-abelian gauge groups. *Commun. Math. Phys.*, **28**, 331–348, 1972. [189](#)
- [49] Doplicher, S., Haag, R., Roberts, J. E. Local observables and particle statistics II. *Commun. Math. Phys.*, **35**, 49–85, 1974. [281](#)
- [50] Doplicher, S., Roberts, J. E. Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics. *Commun. Math. Phys.*, **131**, 51–107, 1990. [6](#), [34](#), [189](#), [190](#), [198](#), [209](#)
- [51] Reed, M., Simon, B. *Methods of Modern Mathematical Physics: Functional analysis*, tomo 1. Academic Press, 1980, 1975. [8](#), [30](#), [125](#)
- [52] Bratteli, O., Robinson, D. W. *Operator algebras and quantum statistical mechanics 2: Equilibrium states. Models in quantum statistical mechanics*. Springer-Verlag Berlin, Germany, 1996. [13](#), [14](#), [17](#), [55](#), [56](#)
- [53] Haag, R., Kastler, D. An Algebraic approach to quantum field theory. *J. Math. Phys.*, **5**, 848–861, 1964. [19](#)
- [54] Sakai, S. *C*-Algebras and W*-Algebras*. Springer-Verlag Berlin Heidelberg, 1998. [23](#)
- [55] Hollands, S., Sanders, K. *Entanglement measures and their properties in quantum field theory*, 2017. [28](#), [42](#), [67](#), [69](#), [246](#)

- [56] Carpi, S., Kawahigashi, Y., Longo, R. Structure and classification of superconformal nets. *Annales Henri Poincaré*, **9**, 1069–1121, 2008. [28](#)
- [57] Halvorson, H., Muger, M. Algebraic quantum field theory. En: J. Butterfield, J. Earman (eds.) Philosophy of physics, págs. 731–864. 2007. [28](#), [38](#), [198](#), [202](#), [209](#), [214](#), [285](#)
- [58] Araki, H. Mathematical Theory of Quantum Fields, tomo 101. Oxford University Press, 2000. [29](#), [198](#), [209](#), [213](#)
- [59] Horuzhy, S. Introduction to algebraic quantum field theory, tomo 19. Netherlands: Kluwer, 1990. [30](#), [31](#), [36](#), [75](#), [110](#), [115](#), [285](#)
- [60] Fredenhagen, K., Hertel, J. Local Algebras of Observables and Point - Like Localized Fields. *Commun. Math. Phys.*, **80**, 555, 1981. [31](#)
- [61] Bostelmann, H. Phase space properties and the short distance structure in quantum field theory. *J. Math. Phys.*, **46**, 052301, 2005. [31](#)
- [62] Longo, R., Xu, F. Relative Entropy in CFT. *Adv. Math.*, **337**, 139–170, 2018. [33](#), [74](#), [147](#), [183](#), [191](#), [215](#), [260](#), [261](#)
- [63] Lindblad, G. Entropy, Information, and Quantum Measurements. *Commun. Math. Phys.*, **33**, 305–322, 1973. [42](#)
- [64] Petz, D. Quantum information theory and quantum statistics. Springer Science & Business Media, 2007. [45](#), [49](#), [61](#), [284](#)
- [65] Ohya, M., Petz, D. Quantum entropy and its use. Springer Science & Business Media, 2004. [45](#), [46](#), [47](#), [48](#), [58](#), [59](#), [284](#)
- [66] Cover, T., Thomas, J. Elements of Information Theory. 2^a ed^{ón}. John Wiley & Sons, Inc., 2006. [45](#)
- [67] Nielsen, M., Chuang, I. Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press, 2011. [45](#), [66](#)
- [68] Jain, R., Radhakrishnan, J., Sen, P. A theorem about relative entropy of quantum states with an application to privacy in quantum communication, 2007. [48](#)
- [69] Vedral, V. The role of relative entropy in quantum information theory. *Rev. Mod. Phys.*, **74**, 197–234, 2002. [49](#)
- [70] Buchholz, D., Fredenhagen, K., D’Antoni, C. The Universal Structure of Local Algebras. *Commun. Math. Phys.*, **111**, 123, 1987. [51](#)

- [71] Haaherup, U. THE STANDARD FORM OF VON NEUMANN ALGEBRAS. *Mathematica Scandinavica*, **37**, 271–283, 1976. [52](#), [57](#)
- [72] Longo, R., Rehren, K.-H. Nets of subfactors. *Rev. Math. Phys.*, **7**, 567–598, 1995. [62](#), [189](#), [218](#), [233](#), [261](#), [307](#), [308](#), [309](#)
- [73] Bell, J. S. On the einstein podolsky rosen paradox. *Physics Physique Fizika*, **1**, 195–200, Nov 1964. URL <https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195>. [63](#)
- [74] Clauser, J. F., Horne, M. A., Shimony, A., Holt, R. A. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, **23**, 880–884, Oct 1969. URL <https://link.aps.org/doi/10.1103/PhysRevLett.23.880>. [63](#), [64](#)
- [75] Freedman, S. J., Clauser, J. F. Experimental Test of Local Hidden-Variable Theories. *Phys. Rev. Lett.*, **28**, 938–941, 1972. [66](#)
- [76] Aspect, A., Grangier, P., Roger, G. Experimental tests of realistic local theories via bell's theorem. *Phys. Rev. Lett.*, **47**, 460–463, Aug 1981. URL <https://link.aps.org/doi/10.1103/PhysRevLett.47.460>.
- [77] Aspect, A., Dalibard, J., Roger, G. Experimental test of bell's inequalities using time-varying analyzers. *Phys. Rev. Lett.*, **49**, 1804–1807, Dec 1982. URL <https://link.aps.org/doi/10.1103/PhysRevLett.49.1804>.
- [78] Tittel, W., Brendel, J., Gisin, B., Herzog, T., Zbinden, H., Gisin, N. Experimental demonstration of quantum correlations over more than 10 km. *Phys. Rev. A*, **57**, 3229–3232, May 1998. URL <https://link.aps.org/doi/10.1103/PhysRevA.57.3229>.
- [79] Tittel, W., Brendel, J., Zbinden, H., Gisin, N. Violation of bell inequalities by photons more than 10 km apart. *Phys. Rev. Lett.*, **81**, 3563–3566, Oct 1998. URL <https://link.aps.org/doi/10.1103/PhysRevLett.81.3563>.
- [80] Weihs, G., Jennewein, T., Simon, C., Weinfurter, H., Zeilinger, A. Violation of bell's inequality under strict einstein locality conditions. *Phys. Rev. Lett.*, **81**, 5039–5043, Dec 1998. URL <https://link.aps.org/doi/10.1103/PhysRevLett.81.5039>.
- [81] Pan, J.-W., Bouwmeester, D., Daniell, M., Weinfurter, H., Zeilinger, A. Experimental test of quantum nonlocality in three-photon greenberger–horne–zeilinger entanglement. *Nature*, **403** (6769), 515–519, 2000. URL <https://doi.org/10.1038/35000514>.

- [82] Rowe, M. A., Kielpinski, D., Meyer, V., Sackett, C. A., Itano, W. M., Monroe, C., *et al.* Experimental violation of a bell's inequality with efficient detection. *Nature*, **409** (6822), 791–794, 2001.
- [83] Gröblacher, S., Paterek, T., Kaltenbaek, R., Brukner, Č., Żukowski, M., Aspelmeyer, M., *et al.* An experimental test of non-local realism. *Nature*, **446** (7138), 871–875, 2007.
- [84] Salart, D., Baas, A., van Houwelingen, J. A. W., Gisin, N., Zbinden, H. Spacelike separation in a bell test assuming gravitationally induced collapses. *Phys. Rev. Lett.*, **100**, 220404, Jun 2008. URL <https://link.aps.org/doi/10.1103/PhysRevLett.100.220404>.
- [85] Ansmann, M., Wang, H., Bialczak, R. C., Hofheinz, M., Lucero, E., Neeley, M., *et al.* Violation of bell's inequality in josephson phase qubits. *Nature*, **461** (7263), 504–506, 2009.
- [86] Giustina, M., Mech, A., Ramelow, S., Wittmann, B., Kofler, J., Beyer, J., *et al.* Bell violation using entangled photons without the fair-sampling assumption. *Nature*, **497**, 227 EP –, 04 2013.
- [87] Larsson, J.-A., Giustina, M., Kofler, J., Wittmann, B., Ursin, R., Ramelow, S. Bell-inequality violation with entangled photons, free of the coincidence-time loophole. *Phys. Rev. A*, **90**, 032107, Sep 2014. URL <https://link.aps.org/doi/10.1103/PhysRevA.90.032107>.
- [88] Christensen, B. G. e. a. Detection-loophole-free test of quantum nonlocality, and applications. *Phys. Rev. Lett.*, **111**, 130406, Sep 2013. URL <https://link.aps.org/doi/10.1103/PhysRevLett.111.130406>.
- [89] Shalm, L. K. e. a. Strong loophole-free test of local realism. *Phys. Rev. Lett.*, **115**, 250402, Dec 2015. URL <https://link.aps.org/doi/10.1103/PhysRevLett.115.250402>. 66
- [90] Einstein, A., Podolsky, B., Rosen, N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, **47**, 777–780, May 1935. URL <https://link.aps.org/doi/10.1103/PhysRev.47.777>. 66
- [91] Takesue, H., Dyer, S. D., Stevens, M. J., Verma, V., Mirin, R. P., Nam, S. W. Quantum teleportation over 100 km of fiber using highly efficient superconducting nanowire single-photon detectors. *Optica*, **2** (10), 832–835, Oct 2015. URL <http://www.osapublishing.org/optica/abstract.cfm?URI=optica-2-10-832>. 66

- [92] Summers, S. J. On the independence of local algebras in quantum field theory. *Rev. Math. Phys.*, **2**, 201–247, 1990. [72](#), [73](#)
- [93] Buchholz, D. Product States for Local Algebras. *Commun. Math. Phys.*, **36**, 287–304, 1974. [73](#)
- [94] Buchholz, D., Wichmann, E. H. Causal Independence and the Energy Level Density of States in Local Quantum Field Theory. *Commun. Math. Phys.*, **106**, 321, 1986. [73](#)
- [95] Davidson, D. R. Modular covariance and the algebraic PCT / spin statistics theorem, 1995. [76](#), [286](#)
- [96] Verch, R., Werner, R. F. Distillability and positivity of partial transposes in general quantum field systems. *Rev. Math. Phys.*, **17**, 545–576, 2005. [77](#)
- [97] Casini, H. Geometric entropy, area and strong subadditivity. *Classical and Quantum Gravity*, **21** (9), 2351, 2004. [80](#)
- [98] Liu, H., Mezei, M. A Refinement of entanglement entropy and the number of degrees of freedom. *JHEP*, **04**, 162, 2013. [81](#)
- [99] Cardy, J. L. Is There a c Theorem in Four-Dimensions? *Phys. Lett.*, **B215**, 749–752, 1988. [83](#)
- [100] Casini, H., Huerta, M., Myers, R. C., Yale, A. Mutual information and the F-theorem. *JHEP*, **10**, 003, 2015. [83](#), [86](#), [88](#), [281](#)
- [101] Jafferis, D. L., Klebanov, I. R., Pufu, S. S., Safdi, B. R. Towards the F-Theorem: N=2 Field Theories on the Three-Sphere. *JHEP*, **06**, 102, 2011. [83](#)
- [102] Myers, R. C., Singh, A. Entanglement Entropy for Singular Surfaces. *JHEP*, **09**, 013, 2012. [84](#)
- [103] Bueno, P., Myers, R. C., Witczak-Krempa, W. Universality of corner entanglement in conformal field theories. *Phys. Rev. Lett.*, **115**, 021602, 2015.
- [104] Elvang, H., Hadjiantonis, M. Exact results for corner contributions to the entanglement entropy and Rényi entropies of free bosons and fermions in 3d. *Phys. Lett.*, **B749**, 383–388, 2015. [84](#)
- [105] Casini, H., Huerta, M. Entanglement entropy in free quantum field theory. *J. Phys.*, **A42**, 504007, 2009. [84](#), [91](#), [94](#), [139](#), [155](#), [183](#)
- [106] Hollands, S. Relative entropy close to the edge, 2018. [88](#)

- [107] Araki, H. On quasifree states of car and bogoliubov automorphisms. *Publ. Res. Inst. Math. Sci. Kyoto*, **6**, 385–442, 1971. [90](#)
- [108] Gaudin, M. Une démonstration simplifiée du théorème de wick en mécanique statistique. *Nuclear Physics*, **15**, 89 – 91, 1960. URL <http://www.sciencedirect.com/science/article/pii/0029558260902856>. [91](#)
- [109] Chung, M.-C., Peschel, I. Density-matrix spectra for two-dimensional quantum systems. *Physical Review B*, **62** (7), 4191–4193, Aug 2000. URL <http://dx.doi.org/10.1103/PhysRevB.62.4191>. [93](#)
- [110] Peschel, I. Calculation of reduced density matrices from correlation functions. *Journal of Physics A: Mathematical and General*, **36** (14), L205–L208, Mar 2003. URL <http://dx.doi.org/10.1088/0305-4470/36/14/101>. [93](#)
- [111] Sorkin, R. D. Expressing entropy globally in terms of (4D) field-correlations. *J. Phys. Conf. Ser.*, **484**, 012004, 2014. [94](#)
- [112] Seeley, R. T. Extension of c^∞ functions defined in a half space. *Proc. Amer. Math. Soc.*, **15**, 625–626, 1964. URL <https://doi.org/10.1090/S0002-9939-1964-0165392-8>. [107](#)
- [113] Araki, H. A lattice of von neumann algebras associated with the quantum theory of a free bose field. *Journal of Mathematical Physics*, **4** (11), 1343–1362, 1963. URL <https://doi.org/10.1063/1.1703912>. [110](#), [113](#), [114](#)
- [114] Araki, H. Von neumann algebras of local observables for free scalar field. *Journal of Mathematical Physics*, **5** (1), 1–13, 1964. URL <https://doi.org/10.1063/1.1704063>. [112](#), [113](#), [115](#), [116](#)
- [115] Guido, D. Modular theory for the von Neumann algebras of Local Quantum Physics. *Contemp. Math.*, **534**, 97–120, 2011. [110](#)
- [116] Camassa, P. Relative Haag duality for the free field in Fock representation. *Annales Henri Poincare*, **8**, 1433–1459, 2007. [116](#)
- [117] Lashkari, N. Modular Hamiltonian for Excited States in Conformal Field Theory. *Phys. Rev. Lett.*, **117** (4), 041601, 2016. [119](#)
- [118] Lashkari, N., Liu, H., Rajagopal, S. Modular Flow of Excited States, 2018. [131](#)
- [119] Ruggiero, P., Tonni, E., Calabrese, P. Entanglement entropy of two disjoint intervals and the recursion formula for conformal blocks. *J. Stat. Mech.*, **1811** (11), 113101, 2018. [133](#)

- [120] Coser, A., Tonni, E., Calabrese, P. Spin structures and entanglement of two disjoint intervals in conformal field theories. *J. Stat. Mech.*, **1605** (5), 053109, 2016.
- [121] Alba, V., Tagliacozzo, L., Calabrese, P. Entanglement entropy of two disjoint intervals in $c=1$ theories. *J. Stat. Mech.*, **1106**, P06012, 2011.
- [122] Calabrese, P., Cardy, J., Tonni, E. Entanglement entropy of two disjoint intervals in conformal field theory. *J. Stat. Mech.*, **0911**, P11001, 2009. [171](#)
- [123] Calabrese, P., Cardy, J., Tonni, E. Entanglement entropy of two disjoint intervals in conformal field theory II. *J. Stat. Mech.*, **1101**, P01021, 2011. [171](#)
- [124] Dupic, T., Estienne, B., Ikhlef, Y. Entanglement entropies of minimal models from null-vectors. *SciPost Phys.*, **4** (6), 031, 2018. [133](#)
- [125] Cardy, J., Tonni, E. Entanglement hamiltonians in two-dimensional conformal field theory. *J. Stat. Mech.*, **1612** (12), 123103, 2016. [133](#)
- [126] Cardy, J., Maloney, A., Maxfield, H. A new handle on three-point coefficients: OPE asymptotics from genus two modular invariance. *JHEP*, **10**, 136, 2017. [133](#), [183](#), [261](#), [288](#)
- [127] Mack, G. Introduction to conformal Invariant quantum field theory in two-dimensions and more dimensions. En: NATO Advanced Summer Institute on Nonperturbative Quantum Field Theory (Cargese Summer Institute) Cargese, France, July 16-30, 1987. 1988. URL <http://www-library.desy.de/cgi-bin/showprep.pl?DESY-88-120>. [134](#)
- [128] Fewster, C. J., Hollands, S. Quantum energy inequalities in two-dimensional conformal field theory. *Rev. Math. Phys.*, **17**, 577, 2005. [134](#), [135](#), [136](#)
- [129] Muskhelishvili, N. I. Singular Integral Equations. Groningen Holland, 1990. [138](#)
- [130] Wong, G. Gluing together Modular flows with free fermions. *JHEP*, **04**, 045, 2019. [147](#)
- [131] V., K. Vertex Algebras for Beginners: Second Edition, tomo 10. American Mathematical Society (University Lecture Series), 1998. [152](#)
- [132] Bischoff, M., Tanimoto, Y. Construction of wedge-local nets of observables through Longo-Witten endomorphisms. II. *Commun. Math. Phys.*, **317**, 667–695, 2013. [152](#)
- [133] Casini, H., Fosco, C. D., Huerta, M. Entanglement and alpha entropies for a massive Dirac field in two dimensions. *J. Stat. Mech.*, **0507**, P07007, 2005. [170](#), [260](#)

- [134] Cardy, J. Some results on the mutual information of disjoint regions in higher dimensions. *Journal of Physics A: Mathematical and Theoretical*, **46** (28), 285402, 2013. URL <http://stacks.iop.org/1751-8121/46/i=28/a=285402>. 171
- [135] Agón, C., Faulkner, T. Quantum Corrections to Holographic Mutual Information. *JHEP*, **08**, 118, 2016. 171
- [136] Cvitković, M., Smith, A.-S., Pande, J. Asymptotic expansions of the hypergeometric function with two large parameters—application to the partition function of a lattice gas in a field of traps. *Journal of Physics A: Mathematical and Theoretical*, **50** (26), 265206, jun 2017. 175
- [137] Kawahigashi, Y., Longo, R., Muger, M. Multiinterval subfactors and modularity of representations in conformal field theory. *Commun. Math. Phys.*, **219**, 631–669, 2001. 183, 261
- [138] Rehren, K.-H., Tedesco, G. Multilocal fermionization. *Lett. Math. Phys.*, **103**, 19–36, 2013. 187, 287
- [139] Buchholz, D., Fredenhagen, K. Locality and the Structure of Particle States. *Commun. Math. Phys.*, **84**, 1, 1982. 190, 234
- [140] Ghosh, S., Soni, R. M., Trivedi, S. P. On The Entanglement Entropy For Gauge Theories. *JHEP*, **09**, 069, 2015. 191
- [141] Soni, R. M., Trivedi, S. P. Entanglement entropy in $(3 + 1)$ -d free $U(1)$ gauge theory. *JHEP*, **02**, 101, 2017.
- [142] Soni, R. M., Trivedi, S. P. Aspects of Entanglement Entropy for Gauge Theories. *JHEP*, **01**, 136, 2016.
- [143] Van Acoleyen, K., Bultinck, N., Haegeman, J., Marien, M., Scholz, V. B., Verstraete, F. The entanglement of distillation for gauge theories. *Phys. Rev. Lett.*, **117** (13), 131602, 2016.
- [144] Donnelly, W., Wall, A. C. Entanglement entropy of electromagnetic edge modes. *Phys. Rev. Lett.*, **114** (11), 111603, 2015.
- [145] Donnelly, W. Decomposition of entanglement entropy in lattice gauge theory. *Phys. Rev.*, **D85**, 085004, 2012.
- [146] Donnelly, W., Wall, A. C. Geometric entropy and edge modes of the electromagnetic field. *Phys. Rev.*, **D94** (10), 104053, 2016.

- [147] Donnelly, W. Entanglement entropy and nonabelian gauge symmetry. *Class. Quant. Grav.*, **31** (21), 214003, 2014.
- [148] Camps, J. Superselection Sectors of Gravitational Subregions. *JHEP*, **01**, 182, 2019. [191](#)
- [149] Casini, H., Huerta, M., Rosabal, J. A. Remarks on entanglement entropy for gauge fields. *Phys. Rev.*, **D89** (8), 085012, 2014. [191](#)
- [150] Xu, F. Some Results On Relative Entropy in Quantum Field Theory, 2018. [191](#), [261](#)
- [151] K. Yu. Dadashyan, S. S. H. Algebras of observables of the free Dirac field. *Theoret. and Math. Phys.*, **36**, 665–675, 1978. [194](#)
- [152] Borchers, H.-J. A remark on a theorem of b. misra. *Comm. Math. Phys.*, **4** (5), 315–323, 1967. URL <https://projecteuclid.org:443/euclid.cmp/1103839939>. [209](#)
- [153] D’Antoni, C. Technical properties of the quasilocal algebra. En: In *Palermo 1989, Proceedings, The algebraic theory of superselection sectors and field theory* 248-258. 1989. [209](#)
- [154] Doplicher, S., Longo, R. Standard and split inclusions of von Neumann algebras. *Invent. Math.*, **75**, 493–536, 1984. [213](#), [279](#)
- [155] Guido, D., Longo, R. An Algebraic spin and statistics theorem. *Commun. Math. Phys.*, **172**, 517, 1995. [221](#)
- [156] Coles, P. J., Berta, M., Tomamichel, M., Wehner, S. Entropic uncertainty relations and their applications. *Reviews of Modern Physics*, **89** (1), 015002, 2017. [233](#)
- [157] Berta, M., Wehner, S., Wilde, M. M. Entropic uncertainty and measurement reversibility. *New Journal of Physics*, **18** (7), 073004, 2016. [233](#)
- [158] Harlow, D., Ooguri, H. Symmetries in quantum field theory and quantum gravity, 2018. [236](#)
- [159] Hamermesh, M. Group theory and its application to physical problems. Courier Corporation, 2012. [239](#)
- [160] Lewkowycz, A., Maldacena, J. Exact results for the entanglement entropy and the energy radiated by a quark. *JHEP*, **05**, 025, 2014. [246](#)
- [161] Dong, S., Fradkin, E., Leigh, R. G., Nowling, S. Topological Entanglement Entropy in Chern-Simons Theories and Quantum Hall Fluids. *JHEP*, **05**, 016, 2008.

- [162] Caputa, P., Simón, J., Štikonas, A., Takayanagi, T. Quantum Entanglement of Localized Excited States at Finite Temperature. *JHEP*, **01**, 102, 2015.
- [163] Nozaki, M., Numasawa, T., Takayanagi, T. Quantum Entanglement of Local Operators in Conformal Field Theories. *Phys. Rev. Lett.*, **112**, 111602, 2014.
- [164] Alcaraz, F. C., Berganza, M. I., Sierra, G. Entanglement of low-energy excitations in Conformal Field Theory. *Phys. Rev. Lett.*, **106**, 201601, 2011.
- [165] Longo, R. Entropy distribution of localised states, 2018. [246](#)
- [166] Roberts, J. E. Spontaneously Broken Gauge Symmetries and Superselection Rules. 1974. [247](#)
- [167] Doplicher, S. Local observables, gauge symmetries and broken symmetries. *Nucl. Phys. Proc. Suppl.*, **18B**, 73–77, 1990.
- [168] Buchholz, D., Doplicher, S., Longo, R., Roberts, J. E. A New look at Goldstone’s theorem. *Rev. Math. Phys.*, **4** (spec01), 49–83, 1992. [247](#), [249](#), [252](#)
- [169] Roberts, J. E. Net Cohomology and Its Applications to Field Theory. págs. 239–268, 1978. [247](#)
- [170] Casini, H., Huerta, M. Entanglement entropy for a Maxwell field: Numerical calculation on a two dimensional lattice. *Phys. Rev.*, **D90** (10), 105013, 2014. [258](#)
- [171] Agon, C. A., Headrick, M., Jafferis, D. L., Kasko, S. Disk entanglement entropy for a Maxwell field. *Phys. Rev.*, **D89** (2), 025018, 2014. [259](#)
- [172] Metlitski, M. A., Grover, T. Entanglement Entropy of Systems with Spontaneously Broken Continuous Symmetry, 2011. [259](#)
- [173] Kallin, A. B., Hastings, M. B., Melko, R. G., Singh, R. R. Anomalies in the entanglement properties of the square-lattice heisenberg model. *Physical Review B*, **84** (16), 165134, 2011. [259](#)
- [174] Xu, F. On Relative Entropy and Global Index, 2018. [261](#)
- [175] Carpi, S., Kawahigashi, Y., Longo, R. Structure and classification of superconformal nets. En: Annales Henri Poincaré, tomo 9, págs. 1069–1121. Springer, 2008. [263](#)
- [176] Gleason, A. M. Measures on the closed subspaces of a hilbert space. *Journal of Mathematics and Mechanics*, **6** (6), 885–893, 1957. [285](#)
- [177] Beltrametti, E. G., Cassinelli, G. The logic of quantum mechanics. D. Reidel Publishing Company, Dordrecht, 1978.

-
- [178] Mittelstaedt, P. Quantum logic. *Compendium of Quantum Physics*. Springer, Berlin, Heidelberg, 2009. [285](#)
- [179] Casini, H., Huerta, M., Magán, J. M., Pontello, D. On the logarithmic coefficient of the entanglement entropy of a Maxwell field, 2019. [289](#)
- [180] Casini, H., Huerta, M., Magán, J. M., Pontello, D. Entanglement entropy and superselection sectors. Part I. Global symmetries. *JHEP*, **2020** (2), 14, 2020. [289](#)
- [181] Evans, L. C. *Partial Differential Equations*. *Graduates Studies in Mathematics*, American Mathematical Society, 2010. [295](#)
- [182] Scholtes, S. *Introduction to Piecewise Differentiable Equations*. Springer-Verlag New York, 2012. [296](#)

Publications

1. *Entanglement entropy and superselection sectors. Part I. Global symmetries*, H. Casini, M. Huerta, J. Magán, D. Pontello, JHEP, 2020 (2), 14 (2020).
2. *Relative entropy for coherent states from Araki formula*, H. Casini, S. Grillo, D. Pontello, Phys. Rev. D 99, 125020 (2019).
3. *Entropy and modular Hamiltonian for a free chiral scalar in two intervals*, R. Arias, H. Casini, M. Huerta, D. Pontello, Phys. Rev. D 98, 125008 (2018).

